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INTRODUCTION TO SU4 AND THE PROPERTIES OF CHARMED HADRONS*

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Introduction

The following is a series of lectures in the Fermilab Academic Lecture Series. The audience consisted mostly of high-energy experimentalists, primarily post-doctoral physicists, advanced graduate students (doing thesis work in conjunction with experiments being carried out at Fermilab), and physicists among the Fermilab staff or User's Group.

The purpose of the lectures, as I defined it, was to enable those attending to better appreciate the theoretical excitement surrounding the idea of a fourth, charmed quark and to provide the foundation necessary to comprehend the rapidly growing literature on this subject. I anticipate that the useful life of these lectures will be short. If charmed hadrons are not experimentally found, interest will quickly wane. If, on the other hand, charm is discovered, these lectures will soon be displaced by far more complete and detailed discussions.

I would like to thank J.K. Walker for organizing the Fermilab Lecture Series, Chris Quigg with whom I learned most of what is contained herein, and Ben Lee for his encouragement and advice. Discussions with other members of the Theoretical Physics Department have been rewarding and enjoyable. I am especially grateful to Bill Francis of the University of Illinois for volunteering to transcribe my chicken tracks into a form suitable for distribution.

However, I am responsible for the final form, and any errors are mine. With regard to errors, I'd like to borrow a word of advice from Ben Lee: "If you find a sign or phase different from mine, you're probably correct. If you reach a qualitatively different conclusion, you're probably wrong." However, I would appreciate being informed of anything which needs correcting.

Martin B. Einhorn
February 1975

CHAPTER I: INTRODUCTION AND SU_2

In the preceding lecture series, Ben Lee has motivated the introduction of a fourth quark from the requirement that there be two "hadronic" doublets for the Weinberg-Salam Theory. Long before gauge theories had been shown to be renormalizable even in the presence of spontaneous symmetry breaking, Bjorken and Glashow [Phys. Letters 11, 255 (1964)] had motivated a fourth quark from the simple requirement of a lepton-hadron symmetry. They observed that there are two doublets of spin one-half leptons, the electron and its neutrino $\begin{pmatrix}\nu_e\\ e^-\end{pmatrix}$ and the muon and its neutrino $\begin{pmatrix}\nu_\mu\\ \mu^-\end{pmatrix}$. Moreover, each has the property that the electric charge of the electron and muon are the same and the neutrinos have charge one unit greater. If we consider three spin one-half quark fields as underlying the observed hadrons, then (from the Gell-Mann-Nishijima formula), we find two quarks, d and s, of equal charge and one quark, u, of charge one unit greater. If we wish to complete an analogy with leptons, we need a fourth quark c whose charge equals that of the u quark. Thus

$$\begin{pmatrix}\nu_e\\ \nu_\mu\\ e^-\\ \mu^-\end{pmatrix} \text{ would be analogous to } \begin{pmatrix}c\\ u\\ d\\ s\end{pmatrix} .$$

An objection to the analogy might be the apparent separate conservation of electronic and muonic lepton number. In addition, the particle spectrum seemed quite remote from any such scheme and, indeed, the idea lay dormant for a few years. Bj and Glashow chose the leptonic weak current to be of the form $L = (\bar{\nu}_e\ \bar{\nu}_\mu)\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}e^-\\ \mu^-\end{pmatrix}$, the equality of diagonal elements expressing electron-muon universality. The hadronic charged current was chosen to be

$$J = \widehat{\underbrace{\bar{c}\ \bar{u}}} \begin{pmatrix} -\sin\theta_C & \cos\theta_C \\ \cos\theta_C & \sin\theta_C \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix} .$$

I shall motivate this form further in Lecture IV. [I've suppressed the space time structure $\gamma^{\mu}(1 - \gamma_5)$ in each case.] A few years later, in a seminal paper, Glashow, Iliopoulous, and Maiani [Phys. Rev. D2, 1285 (1970)] observed that this form makes the algebra of weak current simple,

$$[J, J^{+}] = \bar{c}\,c + \bar{u}\,u - \bar{d}\,d - \bar{s}\,s ,$$

and eliminates induced strangeness-changing neutral currents, to first order in the Fermi coupling G_F.

Without further ado, then, let us consider a world of four quarks, whose strong interactions are invariant under SU_4, except for quark masses. Because they haven't been seen, we expect that the fourth, charmed quark is much heavier than noncharmed quarks. (As it will turn out, we might expect its mass to be about 20 times the strange quark mass.) To understand SU_4, we need to be familiar with SU_3. To understand SU_3, we need SU_2 . . . (fortunately this series rapidly converged).

Before embarking, let me recommend to you several discussions of SU_3 at an introductory level:

P. Carruthers, Introduction to Unitary Symmetry (Interscience, N.Y., 1966).

H. J. Lipkin, Lie Groups for Pedestrians (North Holland Publishing Co., Amsterdam, 1966).

F. E. Low, Symmetries and Elementary Particles (Gordon & Breach, N.Y., 1967).

R. H. Dalitz, lectures delivered at Les Houches Summer School of Theoretical Physics, C. DeWitt and M. Jacob, Editors (Gordon & Breach, N. Y., 1965).

Of these, the latter two employ the tensor formalism I shall use. I would remind you that a systematic guide to the literature may be found in M. Gell-Mann and Y. Ne'eman, The Eightfold Way (W. A. Benjamin, N. Y., 1964). For a general treatment of Lie Groups, see M. Hamermesh, Group Theory (Addison-Wesley, Reading, 1962) and R. E. Behrends et al., "Simple Groups and Strong Interaction Symmetries," Rev. Mod. Phys. **34**, 1 (1962).

I shall begin with SU_2. I assume that you already are well acquainted with the role of isospin symmetry in particle physics, so we can exploit this familiar situation to introduce a formalism and notation which generalizes easily to SU_3.

SU_2 -- Fundamental Representation

SU_2 is defined as that special group of unitary 2×2 matrices with determinant +1. This is the group structure associated with isospin.

The Fundamental Representation, $\underset{\sim}{2}$, of SU_2 will be denoted by the doublet

$$\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix}.$$

Any SU_2 transformation is of the form

$$U(\vec{\alpha}) = e^{i\vec{\alpha}\cdot\vec{G}},$$

but it is sufficient to work with infinitesimal transformations

$$U(\vec{\epsilon}) = 1 + i\vec{\epsilon}\cdot\vec{G}.$$

These G's are the generators of the group. Their form is restricted by the group properties.

$$\text{Unitarity} \Longrightarrow \text{G's are hermitian.}$$

$$\det = 1 \Longrightarrow \text{G's are traceless.}$$

Only three independent 2×2 matrices with these properties exist. It is conventional to choose the generators to be proportional to the usual Pauli matrices.

$$\vec{G} = \frac{1}{2}\vec{\sigma} \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

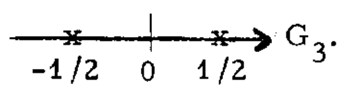
The generators satisfy

$$[G_i, G_j] = i\epsilon_{ijk} G_k.$$

This relation defines the Lie algebra of the group. It is a property of the group, not the particular representation. At most, one of the three generators is diagonal. Again, this is a property of the group. Eigenvectors of G_3 have a fundamental importance independent of particular representation.

The eigenstates will be denoted in the representation above, as

$$\begin{pmatrix} u \\ 0 \end{pmatrix} \text{ for } +1/2 \text{ and } \begin{pmatrix} 0 \\ d \end{pmatrix} \text{ for } -1/2.$$



Another well-known example of an isodoublet is the nucleon pair $N = \begin{pmatrix} p \\ n \end{pmatrix}$ formed by proton p and neutron n.

Next, we want to consider the Conjugate Representation, $\underset{\sim}{2}^*$. This representation transforms as follows

$$\psi^* \to U^* \psi^*$$

where

$$\psi^* = \begin{pmatrix} \psi^{1\dagger} \\ \psi^{2\dagger} \end{pmatrix} \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$

If U is generated by $\vec{G}$, the infinitesimal transformation associated with U^* is

$$U(\vec{\epsilon})^* = 1 - i\vec{\epsilon} \cdot \vec{G}^*.$$

So, U^* is generated by $\vec{H} = -\vec{G}^*$. It is important to note that all eigenvectors of G_3 are still eigenvectors of H_3 but with opposite sign. This suggests that $\underset{\sim}{2}$ and $\underset{\sim}{2}^*$ may be equivalent. They are:

(Proof of equivalence)

Can we find a matrix S such that $S\psi^*$ transforms like ψ?

$$S\psi^* \to U(S\psi^*)$$
$$\psi^* \to (S^{-1} US)\psi^* = U^* \psi^*.$$

An S is needed such that

$$S^{-1} US = U^* \text{ or,}$$
$$S^{-1} \vec{G} S = -\vec{G}^*.$$

S must commute with G_2 while anticommuting with G_1 and G_3. Any matrix proportional to σ_2 will do. Pick

$$S = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

So

$$S\psi^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} \psi_2 \\ -\psi_1 \end{pmatrix} \cong \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix},$$

where the symbol $\cong$ is to be read "transforms like". If the process of complex conjugating is defined to include this nasty minus sign, $\underset{\sim}{2}$ and $\underset{\sim}{2}^*$ will transform with exactly the same equations. This can be done in an invariant way by introducing the totally antisymmetric symbol $\epsilon_{\alpha\beta} = \epsilon^{\alpha\beta}$.

$$\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha} \qquad \alpha, \beta = 1, 2$$

$$\epsilon_{11} = \epsilon_{22} = 0 \qquad \epsilon_{12} = -\epsilon_{21} = 1.$$

This symbol can be shown (Exercise) to be an invariant tensor under SU_2. We have shown above that

$$\psi^\alpha \cong \epsilon^{\alpha\beta}\psi_\beta.$$

There is one other invariant tensor in SU_2, δ^α_β. As usual

$$\delta^1_2 = \delta^2_1 = 0 \text{ and } \delta^1_1 = \delta^2_2 = 1.$$

Exercise: Show that δ^α_β is on invariant tensor and that $\epsilon_{\alpha\gamma}\epsilon^{\gamma\beta} = -\delta^\beta_\alpha$.

Let us discuss another very important representation, the so-called regular representation, $\underset{\sim}{3}$, (isovector). Recall that the generators obey the algebra

$$[G_i, G_j] = i\epsilon_{ijk}G_k.$$

One can use the structure constants to define a representation whose dimension equals the number of generators of the group.

Define $$(T_k)_{ij} \equiv -i\,\epsilon_{ijk}.$$

Then $$[T_k, T_\ell] = +i\,\epsilon_{k\ell m} T_m.$$

Exercise: Prove the above relation.

Hint: $\epsilon_{ijk}\,\epsilon_{\ell mk} = \delta_{i\ell}\,\delta_{jm} - \delta_{im}\,\delta_{j\ell}$.

Aside: Given any Lie Algebra $[G_i, G_j] = i f_{ijk} G_k$, one can always define a representation of dimension equal to the number of generators in a manner similar to the above construction. Just define $(T_k)_{ij} \equiv -i f_{ijk}$. (The proof depends on Jacobi's identity.)

It comes as no surprise that $1/2\,\overline{\psi}\,\sigma_i\,\psi \cong T_i$ (Exercise)

It is useful to define $\sigma_\pm = 1/2\,(\sigma_1 \pm i\,\sigma_2)$, so that

$$\overline{\psi}\,\sigma_+\,\psi = \overline{u}d$$

$$\frac{1}{2}\,\overline{\psi}\,\sigma_3\,\psi = \frac{1}{2}(\overline{u}u - \overline{d}d)$$

$$\overline{\psi}\,\sigma_-\,\psi = \overline{d}u\,.$$

Interpreted as particle states, these three combinations correspond to the three charge states of an isovector multiplet, e.g., π^-, $\pi^0/\sqrt{2}$, π^+, respectively. States in a regular representation can be arranged in matrix form (sometimes convention for calculations) as follows:

$$\overline{\Pi} = \sum_i \frac{1}{2}\,\overline{\psi}\,\sigma_i\,\psi\sigma_i = \frac{1}{2}\,\overline{\psi}\,\vec{\sigma}\psi\cdot\vec{\sigma} = \begin{pmatrix} \frac{1}{2}(u\overline{u}-d\overline{d}) & u\overline{d} \\ d\overline{u} & -\frac{1}{2}(u\overline{u}-d\overline{d}) \end{pmatrix} = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} & \pi^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} \end{pmatrix}.$$

Products of Representations/Irreducible Representations

Consider the tensor $T^{\alpha_1 \cdots \alpha_n} \cong \psi^{\alpha_1} \psi^{\alpha_2} \cdots \psi^{\alpha_n}$ transforming like products of the fundamental representations. It must transform as follows:

$$T^{\alpha_1 \cdots \alpha_n} \rightarrow U^{\alpha_1'}_{\alpha_1} \cdots U^{\alpha_n'}_{\alpha_n} T^{\alpha_1 \cdots \alpha_n} \equiv T^{\alpha_1' \cdots \alpha_n'} .$$

It is easy to see that, if the indices of $T^{\alpha_1 \cdots \alpha_n}$ possess a permutation symmetry, this symmetry is preserved by the transformation.

By definition, an irreducible representation of the group contains no subset of matrices which transform among themselves under the group transformations. Since tensors with a definite permutation symmetry do transform among themselves, it is clear that irreducible representations must have such symmetry properties. This is why a study of the symmetric group is a useful adjunct to a study of groups of linear transformations. (See Hammermesh, Chapters 7 and 10.)

Now apply these ideas to $\underset{\sim}{2} \otimes \underset{\sim}{2}$. The most general element must look like $\psi^\alpha \psi^\beta$. This is not irreducible. Since the irreducible parts must have a definite symmetry, decompose this as follows:

$$\psi^\alpha \psi^\beta = \frac{1}{2} (\psi^\alpha \psi^\beta + \psi^\beta \psi^\alpha) + \frac{1}{2} (\psi^\alpha \psi^\beta - \psi^\beta \psi^\alpha).$$

Often, the fundamental representation of a group is represented by a dot $\cdot$ or box $\square$. We establish the convention that 2 boxes in a horizontal row correspond to a <u>symmetric</u> product of states, while 2 boxes in a vertical column correspond to an <u>antisymmetric</u> combination. Thus the preceding equation may be symbolized as

$$\square \otimes \square = \square\!\square \oplus \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array} .$$

The antisymmetric part of this decomposition is quite simple. α and β must be different to avoid a null result. The remaining two combinations are identical, except for sign:

$\mathbf{1}$: $\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$ $\psi^1 \psi^2 - \psi^2 \psi^1$, e.g., for 2 nucleons, pn - np.

This state corresponds to an isosinglet, I = 0, and can be identified with the deuteron.

The completely symmetric part yields 3 states.

$$\mathbf{3}: \square\!\square \quad \begin{array}{c} \psi^1 \psi^1 \\ \frac{1}{2}(\psi^1 \psi^2 + \psi^2 \psi^1) \\ \psi^2 \psi^2 \end{array} , \text{ e.g., for 2 nucleons, } \begin{array}{c} pp \\ \frac{1}{2}(pn + np) \\ nn \end{array} .$$

These states form an isotriplet, I = 1.

It is established that $\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{3}$. It is important to repeat this for $\mathbf{2} \otimes \mathbf{2}^*$. Nothing new can emerge since $\mathbf{2}^* \cong \mathbf{2}$; however, the notation that results will point the way for some general conclusions. Use $\psi^\alpha \cong \epsilon^{\alpha\beta} \psi_\beta$.

$$\mathbf{1}: \; \psi^1 \psi^2 - \psi^2 \psi^1 \cong -\psi^1 \psi_1 - \psi^2 \psi_2 = -(\psi^\alpha \psi_\alpha), \text{ e.g., } u\bar{u} + d\bar{d}.$$

We have learned the important result that the antisymmetric combination of the two upper indices transforms like the trace of an upper with a lower index. Next consider

$$\mathbf{3}: \; \frac{1}{2}(\psi^\alpha \psi^\beta + \psi^\beta \psi^\alpha) \cong + \frac{1}{2}(\psi^\alpha \epsilon^{\beta\gamma} \psi_\gamma + \psi^\beta \epsilon^{\alpha\gamma} \psi_\gamma).$$

This looks like

$$+\frac{1}{2}\begin{pmatrix} 2\psi^1\psi_2 \\ -\psi^1\psi_1+\psi^2\psi_2 \\ -2\psi^2\psi_1 \end{pmatrix}, \text{ e.g., } \begin{pmatrix} u\bar{d} \\ \frac{1}{2}\ (-u\bar{u}+d\bar{d}) \\ -d\bar{u} \end{pmatrix} \cong \begin{pmatrix} \pi^+ \\ -\pi^o/\sqrt{2} \\ -\pi^- \end{pmatrix}.$$

Theorem: These phases will drive you crazy if you worry about them. My advice: Don't worry.

Since the singlet transformed like the trace, the isotriplet could also have been written

$$(\underset{\sim}{3})^{\alpha}_{\beta} = \psi^{\alpha}\psi_{\beta} - \frac{1}{2}\ \delta^{\alpha}_{\beta}\,(\psi^{\gamma}\psi_{\gamma}).$$

The trace has simply been subtracted in an invariant way.

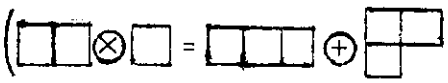
This property of two antisymmetric indices transforming like the trace is much more general. It is easy to see that the antisymmetric piece of any tensor transforms under SU_2 like a tensor of lower rank.

e.g. Suppose a tensor is antisymmetric in any two indices, say α_1 and α_2, then

$$T^{\alpha_1\alpha_2\alpha_3\cdots\alpha_n}_{\beta_1\cdots\beta_m} \cong T^{\gamma\alpha_3\cdots\alpha_n}_{\gamma\beta_1\cdots\beta_m} \equiv V^{\alpha_3\cdots\alpha_n}_{\beta_1\cdots\beta_m}.$$

Since the basis for an irreducible representation must have a definite permutation symmetry, we can exploit this observation to assume, without loss of generality, that the basis of an irreducible representation is symmetric in all its indices.

Now, since the trace of a tensor is a subset of the original tensor which is invariant under group transformations, it must be that, for a tensor to form the basis of an irreducible representation, all traces (contractions of upper with lower indices) must vanish. We combine this with the preceding requirement to conclude that the tensor basis for an irreducible representation of SU_2 must be totally symmetric and traceless. In fact, these conditions are not only necessary but also sufficient, though we won't take the time to prove it here. This important result will be seen to hold also for SU_3 but, alas, not for SU_4.

As a quick example of how we can utilize this result, suppose we combine a triplet with a doublet: $3 \otimes 2 = 4 \oplus 2$. ().

The symmetric part 4 is

$$(\psi^\alpha\psi^\beta\psi^\gamma + \psi^\beta\psi^\gamma\psi^\alpha + \psi^\gamma\psi^\alpha\psi^\beta) + \underbrace{\psi^\beta\psi^\alpha\psi^\gamma + \psi^\alpha\psi^\gamma\psi^\beta + \psi^\gamma\psi^\beta\psi^\alpha}_{\text{just } (\alpha \longleftrightarrow \beta)}.$$

This looks very complicated, but it is not.

$\alpha, \beta, \gamma \neq 2$	uuu	Δ^{++}	
$\alpha, \beta \neq 2$	uud + udu + duu	Δ^{+}	$I = 3/2$
$\alpha \neq 2$	udd + dud + ddu	Δ^{0}	
all = 2	ddd	Δ^{-}	

Reflection Symmetry

Use Condon-Shortley phase conventions so that you can look everything up in tables

$$U(\vec{\alpha})\,|I, I_3\rangle = e^{+i\vec{\alpha}\cdot\vec{I}}\,|I, I_3\rangle$$

$$= D^{I}_{I_3' I_3}(\vec{\alpha})\,|I, I_3'\rangle\,.$$

Reflection corresponds to a rotation of π about the 2-axis. For this the D is simple.

$$D^{I}_{I_3', I_3}(0, \pi, 0) = (-1)^{I+I_3} \delta_{I_3', I_3}.$$

Call this operation R for reflection.

$$R|I, I_3\rangle = (-1)^{I+I_3} |I, -I_3\rangle .$$

Apart from the phase change, it merely flips the third component of isospin.

Charge Conjugation

We want $C|\text{particles}\rangle = (\text{phases})|\text{antiparticles}\rangle$. Now it has already been shown that, if the particle isospinor is $\begin{pmatrix} p \\ n \end{pmatrix}$, the conjugate spinor must be $\begin{pmatrix} \bar{n} \\ -\bar{p} \end{pmatrix}$. So

$$C|I, I_3\rangle = (\text{phases})|I, -I_3\rangle .$$

So C flips the third component of isospin.

G-Parity

If we combine these two operations, we have an operator called G-parity

$$G = CR.$$

This leaves the isospin structure of the states unchanged while changing all particles to antiparticles.

Consider applying G to any pion state. Since both particles and antiparticles lie in the same isospin multiplet

$$G \left| \text{any pion system} \right\rangle = (\text{phase}) \left| \begin{matrix} \text{exactly the} \\ \text{same state} \end{matrix} \right\rangle$$

G-parity will be a good quantum number for this system. One can fix the phases to have

$$G \left| I, I_3 \right\rangle = C_0 (-1)^I \left| I, I_3 \right\rangle$$

where C_0 is the charge conjugation eigenvalue for the neutral member of the multiplet.

Aside: Worth noting that there exists no isospin multiplet which includes particle and antiparticle having half-integral spin. (Carruther's Theorem). See P. A. Carruthers, Spin and Isospin in Particle Physics (Gordon and Breach, N.Y., 1971).

APPENDIX TO LECTURE I

Dimension Counting

Dimension of totally symmetric tensor $T^{\alpha_1 \cdots \alpha_m}$ where α_i can take on N different values is

$$\binom{N+m-1}{m}.$$

So dimension of $T^{\alpha_1 \cdots \alpha_m}_{\beta_1 \cdots \beta_n}$ is

$$\binom{N+m-1}{m}\binom{N+n-1}{n}.$$

The trace conditions give $\binom{N+m-2}{m-1}\binom{N+n-2}{n-1}$ constraints, so the dimension of a totally symmetric traceless tensor is

$$D_n^m = \binom{N+m-2}{m}\binom{N+n-2}{n}\left(1+\frac{m+n}{N-1}\right).$$

For SU_2, $(N = 2)$

$$D_n^m = 1 + m + n.$$

For SU_3, $(N = 3)$

$$D_n^m = (m+1)(n+1)(2+m+n)/2$$

For SU_2 and SU_3, one can obtain these results more simply by essentially just counting, but it is sometimes useful to know the general result for arbitrary **N**.

II. REVIEW OF SU_3

SU_3--Fundamental Representation

SU_3 is defined as that special group of unitary 3×3 matrices with determinant + 1. This is the group structure associated with the combination of isospin and strangeness.

The Fundamental Representation, $\underset{\sim}{3}$, of SU_3 will be denoted by the triplet

$$\psi = \begin{pmatrix} \psi^1 \\ \psi^2 \\ \psi^3 \end{pmatrix} .$$

Much of what was said about SU_2 remains valid for SU_3. Any infinitesimal transformation is of the form

$$U(\vec{\epsilon}) = 1 + i\vec{\epsilon} \cdot \vec{G} \quad .$$

The generators of SU_3 are still hermitian and traceless, but now, of course, they are 3×3 matrices. There exists 8 such matrices that are independent.

As usual some freedom exists in defining these generators. The desire to include isospin as a subgroup fixes three of these to be the Pauli matrices. Define

$$G_i = \frac{1}{2} \lambda_i \; .$$

Then, associating isospin with the 1 and 2 axes suggests

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

These matrices leave the third component of the fundamental triplet untouched.

The other 5 generators mix the isospin components with the new, third component of the triplet. One way to generate the five needed matrices is to exactly imitate the Pauli matrices but connecting components 2 and 3 and then 1 and 3. This is essentially what is done with one important exception. Such a procedure would yield 9 λ-matrices altogether, including 3 diagonal ones. However, these three are not linearly independent. (There can be only two linearly independent 3×3 diagonal, traceless matrices.) It is convenient to retain λ_3 above and to choose the other, λ_8, to be singlet in isospin. It then can be used to distinguish the new quantum number (called hypercharge) carried by the third component. So we define

$$\left.\begin{aligned} \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ \lambda_5 &= \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \end{aligned}\right\} \begin{array}{l} \text{Pauli-type matrices} \\ \text{mixing 1 and 3} \\ \text{components.} \end{array}$$

$$\left.\begin{aligned}\lambda_6 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}\end{aligned}\right\}\text{Pauli-type matrices mixing 2 and 3 components.}$$

$$\left.\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\right\}\text{"new" type diagonal matrix.}$$

These generators satisfy the following algebra

$$\left[G_i, G_j\right] = i\, f_{ijk}\, G_k \quad .$$

All these f_{ijk}'s are just numbers like 1 or $\sqrt{3}/2$ or whatever which can be found tabulated in SU_3 books.

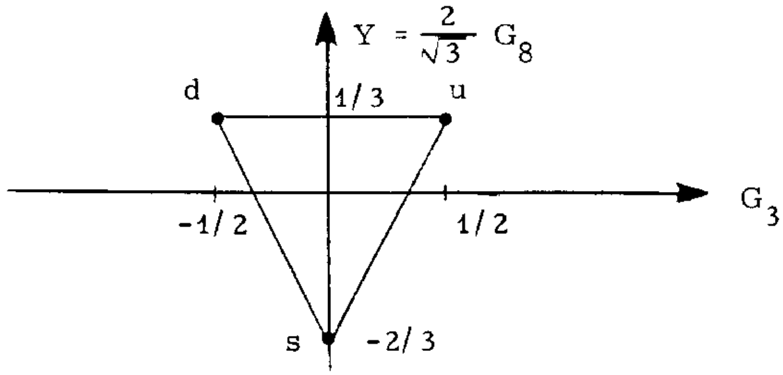
States will be labeled by the eigenvalues of the diagonal matrices.

$$G_3 \begin{pmatrix} u \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} u \\ 0 \\ 0 \end{pmatrix} \quad G_3 \begin{pmatrix} 0 \\ d \\ 0 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 0 \\ d \\ 0 \end{pmatrix} \quad G_3 \begin{pmatrix} 0 \\ 0 \\ s \end{pmatrix} = 0$$

$$G_8 \begin{pmatrix} u \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2\sqrt{3}} \begin{pmatrix} u \\ 0 \\ 0 \end{pmatrix} \quad G_8 \begin{pmatrix} 0 \\ d \\ 0 \end{pmatrix} = \frac{1}{2\sqrt{3}} \begin{pmatrix} 0 \\ d \\ 0 \end{pmatrix} \quad G_8 \begin{pmatrix} 0 \\ 0 \\ s \end{pmatrix} = -\frac{1}{\sqrt{3}} \begin{pmatrix} 0 \\ 0 \\ s \end{pmatrix}$$

A two-dimensional diagram is now needed to display these states. We plot the eigenvalues of G_3 along one axis and G_8 along the other.

It is conventional to plot $Y = \frac{2}{\sqrt{3}} G_8$ rather than G_8 (so that baryons and mesons will have integer values of Y). The fundamental triplet presents the famous triangular pattern



Plots like this of eigenvalues of diagonal generators are called <u>weight diagrams</u>.

Conjugate Representation

Denote the fundamental triplet of the conjugate representation, $\underset{\sim}{3}^*$, by

$$\psi^* = \begin{pmatrix} \psi^{1*} \\ \psi^{2*} \\ \psi^{3} \end{pmatrix} \equiv \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} .$$

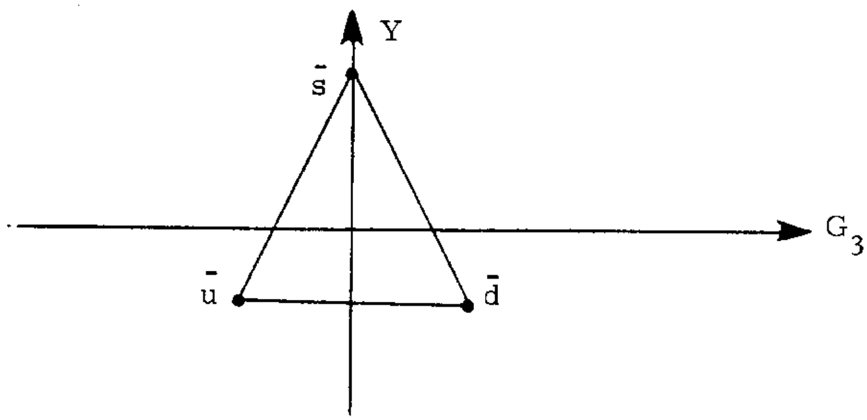
The generators for the conjugate representation will be

$$\vec{H} = -\vec{G}^* .$$

In particular, this means that

$$H_3 = -G_3 \text{ and } H_8 = -G_8 .$$

This is extremely important. H_3 still has the same eigenvalues as G_3. So, if isospin were the only thing around, a similarity transform could be found to demonstrate the equivalence of the regular and conjugate representations. This is, in fact, just what was done for SU_2. Now, however, there is another matrix to worry about, H_8. Since its eigenvalues are completely different from those of G_8, no such procedure can be followed here. (Prove this.) So the conjugate representation is a completely independent representation of SU_3. (In general, a representation is equivalent to its conjugate if and only if its weight diagram is unchanged by inversion.) The weight diagram for $\underset{\sim}{3}^*$ is:



Products of Representations

As in SU_2 there are exactly three invariant tensors in SU_3. These are the completely antisymmetric symbols and the Kronecker delta symbol.

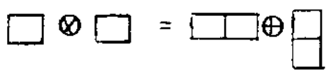
$$\epsilon_{ijk} = \epsilon^{ijk} \; (\epsilon_{123} \equiv 1) \;, \qquad \text{and} \qquad \delta^i_j \;, \quad i,j,k = 1,2,3.$$

(Exercise: Prove that these are invariant tensors).

These will be used in the following discussion.

Now consider the product $\underset{\sim}{3} \otimes \underset{\sim}{3}$. Its general term is of the form $\psi^i\psi^j$, but this is reducible. Attack this by separating the symmetries.

$$\psi^i\psi^j = \underbrace{\tfrac{1}{2}(\psi^i\psi^j + \psi^j\psi^i)}_{\substack{\text{completely}\\ \text{symmetric}}} + \underbrace{\tfrac{1}{2}(\psi^i\psi^j - \psi^j\psi^i)}_{\substack{\text{completely}\\ \text{antisymmetric}}}$$



The antisymmetric part will be investigated first. We shall show that the antisymmetric part of $\psi^i\psi^j$ transforms like $\underset{\sim}{3}^*$.

(Proof.)

The set of objects $\epsilon_{ijk}\psi^i\psi^j$ is identical to the set of objects forming the antisymmetric part of $\psi^i\psi^j$ written above. For example,

$$\begin{aligned}\epsilon_{ij3}\psi^i\psi^j &= \epsilon_{123}\psi^1\psi^2 + \epsilon_{213}\psi^2\psi^1\\ &= \psi^1\psi^2 - \psi^2\psi^1 .\end{aligned}$$

Once the antisymmetric terms are written in this way, it is clear that they must transform like the $\underset{\sim}{3}^*$. For completeness the proof is sketched. Under a transformation U

$$\begin{aligned}\epsilon_{ijk}\psi^i\psi^j &\longrightarrow \underbrace{\epsilon_{ijk}U^i_{i'}U^j_{j'}}\psi^{i'}\psi^{j'}\\ &\longrightarrow \epsilon_{i'j'k'}(U^{-1})^{k'}_{k}\psi^{i'}\psi^{j'}\end{aligned}$$

but $U^{-1} = U^{\dagger}$, so

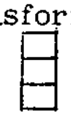
$$\epsilon_{i'j'k'}\, U^{*k}{}_{k'}\, \psi^{i'} \psi^{j'} .$$

So, finally,

$$\epsilon_{ijk}\, \psi^{i}\psi^{j} \rightarrow U^{*k}{}_{k'}\, (\epsilon_{i'j'k'}\psi^{i'} \psi^{j'}) .$$

This is precisely the transformation law for 3^{*}.

This tensor notation developed above is quite useful. It is clear, for example, that

$\epsilon_{ijk}\psi^{i}\psi^{j}\psi^{k}$ must transform like a singlet (diagram ).

An immediate generalization of our little exercise with $\psi^{i}\psi^{j} - \psi^{j}\psi^{i}$ is:

<u>Theorem</u>: If a tensor is antisymmetric in any 2 upper (lower) indices, it is equivalent to a tensor with 2 fewer upper (lower) indices and one additional lower (upper) index.

e.g., If $T^{i_1 i_2 i_3 \dots i_m}_{j_1 j_2 \dots j_m}$ is antisymmetric in i_1 and i_2, then

$$\epsilon_{i_{n+1} i_1 i_2}\, T^{i_1 i_2 \dots i_m}_{j_1 \dots j_n} \cong T^{i_3 i_4 \dots i_m}_{j_1 j_2 \dots j_n j_{n+1}} .$$

What about traces? Well, for the same reasons discussed in SU_2, an irreducible tensor representation must have all traces (contractions) give a null result.

As before, these necessary conditions are sufficient and so, the most general, irreducible tensor representation of SU_3 is a traceless tensor, symmetric in all upper and lower indices.

As we remarked earlier, it has dimension

$$D^m_n = \frac{(m+1)(n+1)(m+n+2)}{2} \quad .$$

Meson Octet

In the quark model of SU_3, the pseudoscalar mesons are identified with quark-antiquark pairs. This leads to $\underset{\sim}{3} \otimes \underset{\sim}{3}^*$. Its general term is $\psi^i\psi_j$, but this is reducible.

One irreducible subgroup is the singlet trace $\psi^k\psi_k$.

If the singlet is removed in an SU_3 invariant way, the result is

$$8: \; \psi^i\psi_j - \frac{1}{3}\,\delta^i_j\psi^k\psi_k \equiv T^i_j \quad .$$

This has zero trace. It is an 8-dimensional irreducible representation, so

$$\underset{\sim}{3} \otimes \underset{\sim}{3} = \underset{\sim}{1} \oplus \underset{\sim}{8}.$$

It is clear that $(T^i_j)^* = T^j_i$, so the conjugate octet is not distinct.

To remove the mystery of the notation, all the states are written out here .

	I_3	Y	Pseudoscalar Particle
$T^1_3 = u\bar{s}$	$+1/2$	1	K^+
$T^2_3 = d\bar{s}$	$-1/2$	1	K^0
$T^1_2 = u\bar{d}$	$+1$	0	π^+

	I_3	Y	Pseudoscalar Particle
$T^2_1 = d\bar{u}$	-1	0	π^-
$T^3_1 = s\bar{u}$	$-1/2$	-1	K^-
$T^3_2 = s\bar{d}$	$+1/2$	-1	$\bar{K}^0$
$T^1_1 = \frac{2}{3}u\bar{u} - \frac{1}{3}(d\bar{d}+s\bar{s})$	0	0	
$T^2_2 = \frac{2}{3}d\bar{d} - \frac{1}{3}(u\bar{u}+s\bar{s})$	0	0	Not all independent since trace must be zero.
$T^3_3 = \frac{2}{3}s\bar{s} - \frac{1}{3}(u\bar{u}+d\bar{d})$	0	0	

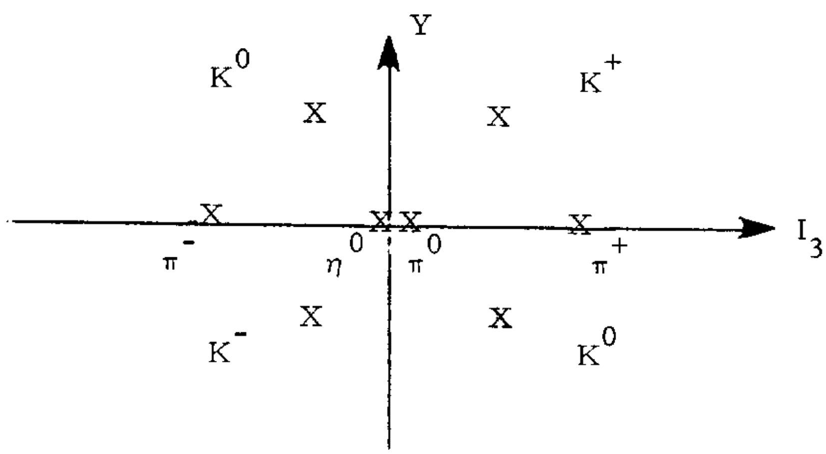
Since the π^0 state is fixed by isospin symmetry, we write

$$\pi^0 = \frac{T^1_1 - T^2_2}{\sqrt{2}} = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$$ and the normalized

orthogonal state is

$$\eta^0 = \sqrt{\frac{3}{2}}\, T^3_3 = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s}) .$$

The π^0 and η^0 are the central members of the octet. The π^0 has total isospin one, the η^0 total isospin zero. These states form the famous picture



There remains a singlet quark-antiquark, the η' or X^0 with

$$\eta' = \frac{1}{\sqrt{3}} (u\bar{u} + d\bar{d} + s\bar{s}).$$

Both η^0 and η' are isosinglets. Because SU_3 is not an exact strong interaction symmetry, it could be that physical states are best described by orthogonal linear combinations of η^0 and η'. It seems, however, for pseudoscalars, such mixing is small. We have suppressed a discussion of the spins of the quarks, but,of course, a complete treatment would require that we take into account the spin one-half nature of the quarks. The above particle names correspond to a pseudoscalar meson. (Because a fermion and its antifermion have opposite intrinsic parity, a $q\bar{q}$ state in an S-wave forms a pseudoscalar.) If, on the other hand, the quarks are combined to form a vector particle, $(\bar{\psi}\gamma^{\mu}\psi)$, the particle names are, in the same order presented above, K^{*+}, K^{*0}, ρ^+, ρ^-, K^{*-}, $\overline{K^{*0}}$, ρ^0, ω_8 and ω_1. I would remind you that the physical states appear to require substantial mixing between ω_8 and ω_1, giving particles called ω and ϕ. They are approximately associated with

$$\omega = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}) = \frac{1}{\sqrt{3}} (\omega_8 + \sqrt{2}\,\omega_1)$$

$$\phi = s\bar{s} = \frac{1}{\sqrt{3}} (\omega_1 - \sqrt{2}\,\omega_8)$$

This is called "canonical mixing" and is motivated by the supposition that SU_3 breaking corresponds to the strange s quark being more massive than u and d (which remain degenerate because of isospin invariance).

I haven't time to go into all the consequences of mixing but I just wanted to refresh your memories about the sort of questions which confuse the identification of states with equal isospin and hypercharge.

Notice that the correct particle identification is made easier by a familiarity with the isospin multiplets . Similarly it will turn out to be useful to know the SU_3 mutliplets for picking out physical states from SU_4 multiplets.

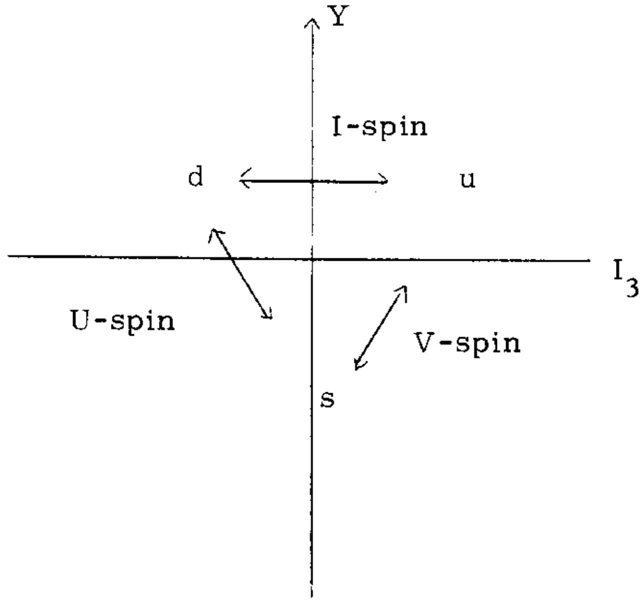
Other SU_2 Subgroups of SU_3

By convention the 1 and 2 indices of the fundamental triplet correspond to an SU_2 subgroup called isospin. Clearly indices 2 and 3 as well as 1 and 3 must form SU_2 subgroups too.

Operations mixing 1 and 2 but leaving 3 unchanged are called I-spin operations.

Operations mixing 2 and 3 but leaving 1 unchanged are called U-spin operations.

Finally, operations mixing 3 and 1 but leaving 2 unchanged are called V-spin operations.



These subgroups can be very useful since SU_2 is somewhat easier to manipulate than SU_3.

Charge Operator/U-Spin

Gell-Mann and Nishijima gave the rule for the charge operator

$$Q = I_3 + \frac{Y}{2} = G_3 + \frac{1}{\sqrt{3}} G_8 .$$

For the quark states which compose the fundamental triplet

u	$Q = 2/3$
d	$Q = -1/3$
s	$Q = -1/3$

Note that U-spin operators mix d- and s-quarks only; ie., U-spin operators only connect states with equal charge. At this point it is interesting to write down the electromagnetic current in the simplest quark model--

$$J_\mu = \sum_j e_j \bar{\psi}_j \gamma_\mu \psi^j \quad .$$

Ignoring the space-time structure

$$J = \frac{2}{3}\bar{u}u - \frac{1}{3}(\bar{d}d + \bar{s}s)$$

$\bar{d}d + \bar{s}s$ is a U-spin singlet (compare to $\bar{d}d + \bar{u}u$ in isospin). Also $\bar{u}u$ is a U-spin singlet.

The electromagnetic current, then, transforms like a U-spin singlet. This has an interesting consequence. Just as G-parity was defined for isospin, so G_U -parity can be defined in the U-spin subspace:

$$G_U \equiv CR_U \equiv Ce^{-i\pi U_2}$$

Applied to members of the U-spin triplet contained in the SU_3 octet of mesons gives, for example,

$$G_U \mid K^0 > = - \mid K^0 >$$

just like

$$G_I \mid \pi^+ > = - \mid \pi^+ > \quad .$$

Now one can show that, in the limit of exact SU_3 symmetry, a photon cannot produce neutral kaon pairs:

Consider $e\bar{e} \to \gamma^* \to K^0\overline{K^0}$. This process involves the matrix element of the electromagnetic current between the vacuum and the $K^0\overline{K^0}$ state.

$$<0 \mid J \mid K^0\overline{K^0}> = <0 \mid G_U^{-1}\, G_U J\, G_U^{-1}\, G_U \mid K^0\overline{K^0}>$$

If the vacuum is symmetric, then

$$\langle 0 | J | K^0 \overline{K^0} \rangle = \langle 0 | G_U \, J \, G_U^{-1} \, G_U | K^0 \overline{K^0} \rangle \quad .$$

Since the current is a singlet under U-spin, but odd under charge conjugation,

$$G_U \, J \, G_U^{-1} = -J \quad .$$

Using G_U-parity symmetry, $G_U | K^0 \overline{K^0} \rangle = | K^0 \overline{K^0} \rangle$, so that

$$\langle 0 | J | K^0 \overline{K^0} \rangle = - \langle 0 | J | K^0 \overline{K^0} \rangle \; .$$

Thus, this matrix element vanishes in the limit of SU_3 symmetry. (Later, we will find similar applications of SU_2 subgroups of SU_4 representations.)

Regular Representation of SU_3: $\underset{\sim}{8}_A$

As indicated in Chapter I, a representation of a Lie Algebra can awalys be constructed from the structure constants by defining

$$(F_k)_{ij} = -i \, f_{ijk} \quad .$$

We thus obtain eight 8 by 8 matrices such that $[F_i, F_j] = i \, f_{ijk} F_k$ (Exercise: Prove this.)

It will come as no surprise to you that

$$\frac{1}{2} \bar{\psi} \lambda_k \psi \cong F_k \qquad \text{(Prove this, too)}$$

More explicitly, the left-hand side is

$$\frac{1}{2} (\lambda_k)_{ij} \psi_i \, \psi^j = \frac{1}{2} (\lambda_k)_{ij} \, T_i^j$$

where $T_i^j = \psi_i \psi^j - \frac{1}{3} \, \delta_i^j \, (\psi_k \psi^k)$ is the octet introduced previously.

(The singlet comes for free since $\mathrm{Tr}\,\lambda_k = 0$.) Thus the l.h.s. is simply a rearrangement of the octet, and the 8-dimensional regular representation is equivalent to this octet T_i^j. As we illustrated with SU_2, we can represent the regular representation in matrix form

$$\mathscr{P} = \frac{1}{2} \sum_i \bar{\psi} \lambda_i \psi \lambda_i \equiv \frac{1}{2} \bar{\psi} \vec{\lambda} \psi \cdot \vec{\lambda} .$$

$$\mathscr{P} = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^0 & K^0 & -\frac{2}{\sqrt{6}}\eta \end{pmatrix}$$

(Exercise: Discuss the transformation of $\mathscr{P}$ under an SU_3 transformation $U = e^{\frac{i}{2}\vec{\alpha}\cdot\vec{\lambda}}$. What is UPU^{-1}? (Hint: take α to be infinitesimal.))

So far, we have described the mesons in terms of their quark-antiquark content. Baryons can also be arranged in SU_3 multiplets such as the octets (containing the nucleon N) and decuplet (containing the (3, 3) resonances Δ). As you know, these particles can be represented as if they were built of three quarks, so we must analyze $\underset{\sim}{3} \otimes \underset{\sim}{3} \otimes \underset{\sim}{3}$.

Baryons

The terms have the general form $\psi^i\psi^j\psi^k$, but this is reducible. Part of the work is already done since we discussed $\mathbf{3}\otimes\mathbf{3}$ earlier.

$$\mathbf{3}\otimes\mathbf{3}\otimes\mathbf{3} = (\mathbf{6}\oplus\mathbf{3}^*)\otimes\mathbf{3} = \mathbf{6}\otimes\mathbf{3}\oplus\mathbf{3}^*\otimes\mathbf{3}$$

Consider $\mathbf{3}^*\otimes\mathbf{3}$ first. These nine states are already known to form $\mathbf{8}\oplus\mathbf{1}$, but an explicit representation is desired here. The set of objects $\mathbf{3}^*$ has the form

$$\mathbf{3}^* : \psi^i\psi^j - \psi^j\psi^i \equiv (\text{tensor})^{[i,j]}.$$

As previously shown, this set can be written as $\epsilon_{ijk}\psi^i\psi^j$ which clearly demonstrates that they transform like ψ_k. Now $\mathbf{3}^*\otimes\mathbf{3}$ was reduced earlier as follows:

$$\psi_k\psi^\ell = \underbrace{(\psi_k\psi^\ell - \tfrac{1}{3}\,\delta_k^{\ \ell}[\psi_m\psi^m])}_{\mathbf{8}} + \underbrace{\tfrac{1}{3}\,\delta_k^{\ \ell}\,[\psi_m\psi^m]}_{\mathbf{1}}$$

We can use the fact that $\psi_k \cong \epsilon_{ijk}\psi^i\psi^j$ to convert this quark-antiquark decomposition into one involving 3 quarks. For example, consider the singlet

$$(1)\ \psi_k\psi^k \cong \epsilon_{ijk}\psi^i\psi^j\psi^k \quad \left(\text{denoted diagramatically as} \quad \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array}\right)$$

Similarly, the octet

$$\psi_k\psi^\ell - \tfrac{1}{3}\delta_k^{\ \ell}(\psi_m\psi^m) \cong \epsilon_{ijk}\psi^i\psi^j\psi^\ell - \tfrac{1}{3}\delta_k^{\ \ell}(1)$$

or, applying ϵ^{mnk} to this, we get an alternate form

$$(\mathbf{8}_A)^{mn\ell} \equiv (\psi^m\psi^n - \psi^n\psi^m)\psi^\ell - \tfrac{1}{3}\,\epsilon^{mn\ell}(\epsilon_{ijk}\psi^i\psi^j\psi^k).$$

(We used $\epsilon^{mnk}\epsilon_{ijk} = \delta_i^{\ m}\delta_j^{\ n} - \delta_j^{\ n}\delta_i^{\ n}$.)

$$[\psi^m, \psi^n]\psi^\ell = (\underset{\sim}{8})^{[m,n]\ell} + \frac{1}{3}\epsilon^{mn\ell}(\underset{\sim}{1})$$

This octet has a mixed symmetry. By definition, it is antisymmetric in the first two quarks.

Now consider $\underset{\sim}{6} \otimes \underset{\sim}{3}$. There must be 18 states here. It is important to recall that $\underset{\sim}{6}$ is the symmetric combination

$$\underset{\sim}{6} : \psi^i\psi^j + \psi^j\psi^i \equiv (\text{tensor})^{\{i,j\}}.$$

One irreducible representation must be the totally symmetric combination. This is easy to write down

$$\underset{\sim}{10} : \psi^i\psi^j\psi^k + \psi^j\psi^k\psi^i + \psi^k\psi^i\psi^j + \\ + \psi^j\psi^i\psi^k + \psi^i\psi^k\psi^j + \psi^k\psi^j\psi^i.$$

As indicated, there are 10 states here. Denote these by the symbol

$$(\underset{\sim}{10})^{ijk}.$$

Eight states remain to be discovered. These can be constructed very simply using our tensor notation. The symmetry in i and j is fixed by the $\underline{6}$; however, there is no reason why the third index cannot be made antisymmetric with either of these. In tensor notation this reads

$$\epsilon_{\ell jk}(\psi^i\psi^j + \psi^j\psi^i)\psi^k = (\text{tensor})_\ell^{\ i}.$$

This combination is already traceless upon contraction of its upper index with its lower index. This type of tensor is already known to transform as an $\underset{\sim}{8}$. Denote these states by the symbol $(\underset{\sim}{8}_S)_\ell^{\ i}$. The desired decomposition is therefore some linear combination

$$\underset{\sim}{6} \otimes \underset{\sim}{3} : \{\psi^i, \psi^j\}\psi^k = A_{10}(\underset{\sim}{10})^{ijk} + A_8\left[\epsilon^{jk\ell}(\underset{\sim}{8}_S)_\ell^{\ i} + \epsilon^{ik\ell}(\underset{\sim}{8}_S)_\ell^{\ j}\right].$$

The constants A_{10} and A_8 can be shown to be

$$A_{10} = \frac{1}{3} \qquad A_8 = \frac{1}{3}. \qquad \text{(Exercise: show this.)}$$

In summary,

$$\underset{\sim}{3} \otimes \underset{\sim}{3} \otimes \underset{\sim}{3} = \underset{\sim}{10} \oplus \underset{\sim}{8}_A \oplus \underset{\sim}{8}_S \oplus \underset{\sim}{1}.$$

The Baryon Decuplet $\underset{\sim}{10}$

The easiest way to identify the states is to pick out the SU_2 subgroups. These are obtained by setting 0, 1, 2, or 3 indices equal to 3.

$i, j, k \neq 3$:

hypercharge: $Y = +1$

$$\begin{cases} 10^{111} = \{uuu\} \propto uuu & \Delta^{++} \\ 10^{112} = \{uud\} \propto udd + dud + ddu & \Delta^{+} \\ 10^{122} = \{udd\} \propto udd + dud + ddu & \Delta^{o} \\ 10^{222} = \{ddd\} \propto ddd & \Delta^{-} \end{cases}$$

let one index be 3 $Y = 0$

$$\begin{cases} 10^{113} = \{uus\} & Y^{*+} \\ 10^{123} = \{uds\} & Y^{*o} \\ 10^{223} = \{dds\} & Y^{*-} \end{cases}$$

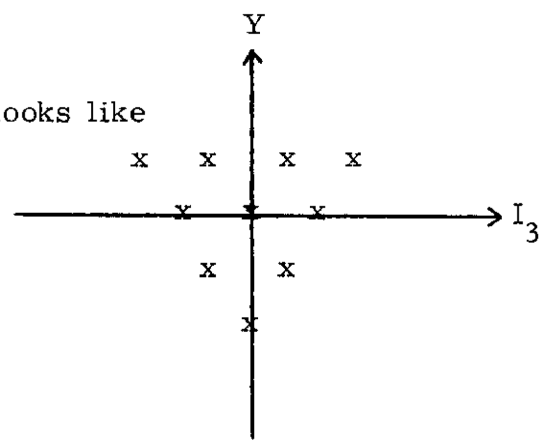
let two indices be 3 $Y = -1$

$$\begin{cases} 10^{133} = \{uss\} & \Xi^{*o} \\ 10^{233} = \{dss\} & \Xi^{*-} \end{cases}$$

let all indices be 3 $Y = -2$

$$10^{333} = \{sss\} \qquad \Omega^{-}$$

The weight diagram looks like



The Baryon Octet

In SU_3 there are two distinct octets within $\underset{\sim}{3} \otimes \underset{\sim}{3} \otimes \underset{\sim}{3}$. The one from $\underset{\sim}{6} \otimes \underset{\sim}{3}$ is symmetric in the first two indices; the one from $\underset{\sim}{3}^* \otimes \underset{\sim}{3}$ antisymmetric in the first two. Both have mixed symmetry properties. One might expect two such octets of baryons to appear in nature. This turns out not to be the case. The reason is the Pauli principle. To see this is a bit complicated, so let me first indicate how the Pauli principle can work to select a subset of states as physically allowed. Suppose we ignore spin altogether and the "quarks" were bosons. Then the "baryon" (spin zero) wave function would have to by symmetric under exchange of any two "quarks." Suppose the wave function were totally symmetric in space, i.e., symmetric under position interchange. Then it would have to be symmetric in SU_3, so its SU_3 part would look like

$$(\underset{\sim}{8}_S)^{ijk} + (\underset{\sim}{8}_S)^{jki} + (\underset{\sim}{8}_S)^{kij}.$$

$\underset{\sim}{8}_A$ simply could not be used since it is antisymmetric. Now how does it work with spinor quarks? In the non-relativistic quark model, the lowest lying states are symmetric in space (no orbital excitation). The correct spectrum is obtained only by requiring the baryon wave function to be totally symmetric in space $\times$ spin $\times$ SU_3. [This used to be called "quark statistics" and nowadays is described by supposing the quarks carry another quantum number (called color) in which the wave function is totally antisymmetric. This ad hoc device preserves the Pauli principle, but is not without physical consequences because the number of degrees of freedom of the fundamental constitutents has been increased.] To form a spin 1/2 baryon requires a spin wave function of mixed symmetry. To obtain an overall symmetric spin $\times$ SU_3 state requires a unique combination of $\underset{\sim}{8}_S$ and $\underset{\sim}{8}_A$. Thus, as in the boson case, we are led to only 8 physically allowed states. I apologize for this long digression, but I do not wish to take the even longer path of introducing spin wave functions and discussing SU_6 (and eventually SU_8). If you wish to see the explicit wave function, see, e.g., J. J. J. Kokkedee, The Quark Model (W. A. Benjamin, N.Y., 1969). (You may be concerned that the language used is non-relativistic, but the result that the Pauli principle selects a unique octet is presumably covariant. I do not know where this has been discussed in all its painful detail.)

But now you can begin to appreciate why the decuplet has spin 3/2. We identified it with a totally symmetric SU_3 state, $(\underset{\sim}{10})^{ijk}$, so, assuming no orbital excitation, it will have a symmetric space piece, so it must also be spin-symmetric. This requires that the spin of the 3 quarks all line up $\uparrow\uparrow\uparrow$ to give spin 3/2.

Returning to the baryon octet, where the discussion began, let us now display the 8 states with their associated quark content.

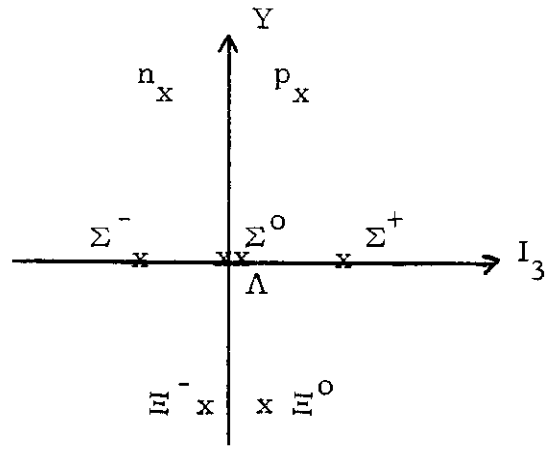
$$\text{udd} - n \qquad\qquad \text{uud} - p$$

$$\text{dds} - \Sigma^- \qquad\qquad \frac{1}{\sqrt{2}}(\text{ud} + \text{du})\text{s} - \Sigma^0 \qquad\qquad \text{uus} - \Sigma^+$$

$$\frac{1}{\sqrt{2}}(\text{ud} - \text{du})\text{s} - \Lambda^0$$

$$\text{dss} - \Xi^- \qquad\qquad \text{uss} - \Xi^0$$

or, on a weight diagram,



Later, we shall be discussing non-leptonic decays of strange and charmed particles. To do so, we need to know how multiparticle states transform under SU_3, e.g., what are the possible SU_3 invariant combinations of two mesons, each of which lies in an octet? Consequently, the decomposition of the product $\underset{\sim}{8} \otimes \underset{\sim}{8}$ will first be reviewed. This will serve, one more time, to illustrate the facility of this tensor formalism.

$\underset{\sim}{8} \otimes \underset{\sim}{8}$ contains 64 states. Since each octet transforms like $T_k^{\ i}$, $\underset{\sim}{8} \otimes \underset{\sim}{8}$ must look something like

$$\underset{\sim}{8} \otimes \underset{\sim}{8} \sim T_k^{\ i}\, T_\ell^{\ j} \equiv V_{k\ell}^{\ ij} \, .$$

Now pick out the irreducible representations. One will be the totally symmetric combination which is traceless. Define

$$S_{k\ell}^{\ ij} \equiv V_{k\ell}^{\ ij} + V_{k\ell}^{\ ji} + V_{\ell k}^{\ ij} + V_{\ell k}^{\ ji} \, .$$

This is certainly symmetric in both upper and lower indices, but not traceless. The correct totally symmetric, traceless irreducible tensor is

$$(\underset{\sim}{27})_{k\ell}^{\ ij} \equiv S_{k\ell}^{\ ij} - (\text{all traces}) .$$

The notation should be obvious. There are 27 independent states here.

(Exercise: Work out the state explicitly. Prove there are 27 states.)

What other tensors are possible? The invariant tensor, ϵ^{ijk}, can be used to isolate the antisymmetric combination of lower indices.

$$W^{kij} \equiv \epsilon^{k\ell m} V_{\ell m}^{\ ij}.$$

This will be irreducible if it is made symmetric in its indices.

$$(\underset{\sim}{10})^{kij} \equiv W^{kij} + W^{ijk} + W^{jki} + W^{kji} + W^{jik} + W^{ikj}.$$

A tensor completely symmetric in three upper indices is known already (from the work on $\underset{\sim}{6} \otimes \underset{\sim}{3}$) to be a $\underset{\sim}{10}$.

If the antisymmetric combinations of upper indices are isolated,

$$\tilde{W}_{k\ell m} \equiv \epsilon_{kij} V_{\ell m}^{\ ij}$$

when made symmetric in the three lower indices exactly as above, the $\underline{10}^*$ results $-(\underset{\sim}{10}^*)_{jk\ell}$. Note that $V_{i\ell}^{\ ij} = 0$ and $V_{kj}^{\ ij} = 0$, because the octets are traceless. However, another tensor can be formed by contracting an upper and a lower index, $V_{ki}^{\ ij}$. This will represent an octet if the trace is removed in the usual way

$$(\underset{\sim}{8}_1)_k^{\ j} \equiv V_{ki}^{\ ij} - \frac{1}{3} \delta_k^{\ j} (V_{\ell i}^{\ i\ell}).$$

A second octet can be formed by contracting the two other indices.

$$(\underset{\sim}{8}_2)_\ell^{\ i} \equiv V_{j\ell}^{\ ij} - \frac{1}{3} \delta_\ell^{\ i} (V_{ki}^{\ ik}).$$

One often changes the basis somewhat by defining the following symmetric and antisymmetric octets:

$$8_S = 8_1 + 8_2 \ \text{ and } \ 8_A = 8_1 - 8_2.$$

So far 63 of the 64 states have been detailed. The singlet that remains is just the trace that we have been removing above.

$$(\underset{\sim}{1}) \equiv V^{ij}_{ji}.$$

This completes the reduction of $\underset{\sim}{8} \otimes \underset{\sim}{8}$. Explicitly

$$T_k^{\ i} T_\ell^{\ j} = A_{27} (\underset{\sim}{27})^{ij}_{k\ell} + A_{10} \epsilon_{k\ell m} (\underset{\sim}{10})^{mij} + A_{10^*} \epsilon^{ijm} (\underset{\sim}{10^*})_{mk\ell}$$

$$+ A_{8_S} \left[\delta_\ell^{\ i} (\underset{\sim}{8}_S)_k^{\ j} + \delta_k^{\ j} (\underset{\sim}{8}_S)_\ell^{\ i} - \frac{2}{3} \delta_k^{\ i} (\underset{\sim}{8}_S)_\ell^{\ j} - \frac{2}{3} \delta_\ell^{\ j} (\underset{\sim}{8}_S)_k^{\ i} \right]$$

$$+ A_{8_A} \left[\delta_\ell^{\ i} (\underset{\sim}{8}_A)_k^{\ j} - \delta_k^{\ j} (\underset{\sim}{8}_A)_\ell^{\ i} \right] + A_1 \left[\delta_\ell^{\ i} \delta_k^{\ j} - \frac{1}{3} \delta_k^{\ i} \delta_\ell^{\ j} \right] (\underset{\sim}{1}).$$

(Exercise: Work out the values of the coefficients A_r.)

As an example, consider the strong, SU_3 invariant decay of a vector meson into two pseudoscalars, e.g., $\rho \to 2\pi$. Now a vector cannot couple to both of the octets $\underset{\sim}{8}_S$ and $\underset{\sim}{8}_A$. Again, this is due to the Pauli principle. The two pions from the decay have to be in a state of total angular momentum 1. Since they are spinless, this requires their space wave function to be the antisymmetric $\ell = 1$ state. To maintain Bose statistics, the SU_3 part must also be antisymmetric. If we call the vectors $V_j^{\ i}$ and the pseudoscalars $P_j^{\ i}$, then the proper invariant coupling is simply proportional to

$$V_j^{\ i} (\underset{\sim}{8}_A)_i^{\ j},$$

where $(\underset{\sim}{8}_A)_i^{\ j} \equiv P_k^{\ j} P_i^{\ k} - P_i^{\ k} P_k^{\ j}$. If you will write this out, you will find all the Clebsch-Gordon coefficients for $\underset{\sim}{8} \otimes \underset{\sim}{8} \to \underset{\sim}{8}$. The identification with physical states requires taking into account the nonet structure, i.e., mixing between the octet and singlet. We leave this as an exercise.

This concludes our review of SU_3. The subjects and examples treated have been chosen with a view toward later applications. As you will see, we will need most of the machinery developed so far. Now that we are warmed up, let's take on one more quark. (For those interested, I append a discussion of mass splitting.)

APPENDIX TO CHAPTER II

Mass Splitting

It is commonly supposed that the fundamental interaction between quarks (say, via gluons) is SU_3 invariant and that the source of the observed violations of SU_3 invariance is due to the strange quark s being intrinsically heavier than the isodoublet u and d. In a quark Lagrangian, the mass term would look like

$$M = m_0(\bar{u}u + \bar{d}d) + m_s(\bar{s}s)$$

$$= m(\psi_i\psi^i) + \Delta m\,[\psi_3\psi^3 - \tfrac{1}{3}(\psi_i\psi^i)]$$

where $m = \frac{2}{3}\,m_0 + \frac{1}{3}\,m_s$ is the average mass and $\Delta m = m_s - m_0$ is the mass splitting of the strange quark.

More abstractly,

$$M = m(\underset{\sim}{1}) + \Delta m(\underset{\sim}{8})^3_3,$$

that is, the mass term transforms as the sum of an SU_3 singlet plus the $^3_3$ component of an octet. Now imagine calculating the first order correction to the mass matrix, perturbing in $\Delta m(\underset{\sim}{8})^3_3$. The general form of this correction for an arbitrary multiplet is called the Gell-Mann-Okubo formula, whose derivation will be discussed below. For many cases, one can obtain mass relations simply by observing that M can also be written as

$$M = A + B(U_3 + \tfrac{1}{2}\,Q)$$

where U_3 is the third component of U-spin and Q is the charge operator. Consequently, within a given U-spin multiplet, there is an equal splitting between adjacent states. For example, in the decuplet, $(\Omega^-, \Xi^{*-}, Y^{*-}, \Delta^-)$ form a U = 3/2 multiplet. The above observation tells us that

$$M(\Omega^-) - M(\Xi^*) = M(\Xi^*) - M(Y^*) = M(Y^*) - M(\Delta).$$

Similarly, for the baryon octet, (Ξ^0, Σ_U^0, n) form a U = 1 multiplet, where $\Sigma_U^0 \equiv \frac{\sqrt{3}}{2}\Lambda - \frac{1}{2}\Sigma^0$. Hence, $M(N) - M(\Sigma_U^0) = M(\Sigma_U^0) - M(\Xi^0)$ or $\frac{1}{2}[M(N) + M(\Xi^0)] = M(\Sigma_U^0) = \frac{1}{4}[3M(\Lambda) + M(\Sigma^0)]$. For mesons, for some reason, the formulas work much better if you use mass-squared. For the pseudoscalar octet, the corresponding relation would be

$$M(K)^2 = \frac{1}{4}[3M(\eta)^2 + M(\pi)^2].$$

[K and $\overline{K}$ have same mass.]

So, for most cases of interest, nothing more than a knowledge of SU_2 is required. However, let us consider the general problem of finding the general form of

$$\left\langle T^{i_1'i_2'\cdots i_n'}_{j_1'\cdots j_m'} \middle| (\underset{\sim}{8})^3_3 \middle| T^{i_1 i_2\cdots i_n}_{j_1\cdots j_m} \right\rangle,$$

where $\left| T^{i_1\cdots i_n}_{j_1\cdots j_m} \right\rangle$ represent the states of an irreducible representation of dimension $D_m^n = \frac{1}{2}(m+1)(n+1)(m+n+2)$. This is just some linear combination of $\underset{\sim}{D}_m^n \otimes \underset{\sim}{D}_m^{n*}$ which transforms like an octet. There are clearly only two possible ways to pick out octets from the product

$$T^{i_1 \cdots i_n}_{j_1 \cdots j_m} \left(T^{i_1' \cdots i_n'}_{j_1' \cdots j_m'} \right)^* \equiv T^{i_1 \cdots i_n}_{j_1 \cdots j_m} (T^*)^{j_1' \cdots j_m'}_{i_1' \cdots i_n'} \text{ , viz,}$$

$$(\underset{\sim}{8}_1)^{i_1}_{i_1'} \equiv T^{i_1 i_2 \cdots i_n}_{j_1 \cdots j_m} (T^*)^{j_1 \cdots j_m}_{i_1' i_2 \cdots i_n} - \frac{1}{3}\, \delta^{i_1}_{i_1'} (\underset{\sim}{1}), \text{ and}$$

$$(\underset{\sim}{8}_2)^{j_1'}_{j_1} \equiv T^{i_1 \cdots i_n}_{j_1 j_2 \cdots j_m} (T^*)^{j_1' j_2 \cdots j_m}_{i_1 \cdots i_n} - \frac{1}{3}\, \delta^{j_1'}_{j_1} (\underset{\sim}{1}),$$

where $$(\underset{\sim}{1}) \equiv T^{i_1 \cdots i_n}_{j_1 \cdots j_m} (T^*)^{j_1 \cdots j_m}_{i_1 \cdots i_n} .$$

As usual, it is frequently more convenient to work with the symmetric and antisymmetric combinations

$$\underset{\sim}{8}_S = \underset{\sim}{8}_1 + \underset{\sim}{8}_2 \text{ and } \underset{\sim}{8}_A = \underset{\sim}{8}_1 - \underset{\sim}{8}_2 .$$

The general form of the mass splitting is therefore

$$\Delta M = A(\underset{\sim}{8}_A)^3_3 + S(\underset{\sim}{8}_S)^3_3,$$

with A and S arbitrary constants. There are two exceptions to this general form. (1) If the particle multiplet is <u>triangular</u>, i.e., has only upper or only lower indices, then we can form only one such octet. For example, for the $3/2^+$ decuplet $(\underset{\sim}{10})^{ijk}$, only $\underset{\sim}{8}_1$ can be defined. (2) If the particle multiplet is self-conjugate (i.e., contains both particle and antiparticle), then $A = 0$. This is again a consequence of the Generalized Pauli Principle. This occurs for all meson multiplets. In each of these two exceptional cases, the mass splitting will depend on only a single parameter, so the results obtained above on the basis of U-spin are completely general.

The above form, although correct and simple to derive, is not so transparent as another formulation. The point is that particle states are more conveniently labeled by their eigenvalues of hypercharge Y, isospin I, and the third component of isospin I_3 or charge. (Since the strong interactions are isospin invariant, the mass is independent of I_3.) We would like to obtain a form which expresses ΔM directly in terms of Y and I. Unfortunately, I know of no simple way to convert the formula above for ΔM into an expression involving Y and I. (Here's a good homework problem!) The simplest way I know is to use the Wigner-Eckart theorem to say that matrix elements of $(\underset{\sim}{8})^3_3$ are proportional to the two independent operators having this transformation property which can be formed from the generators. Instead of numbering the generators G_i, $(i = 1, \cdots 8)$, we define an alternative representation of the 8 generators A^j_k $(j, k = 1, 2, 3)$ with $A^j_j = 0$. In the fundamental representation, these generators are defined to have matrix elements

$$(A^j_k)_{mn} = \delta_{jm}\delta_{kn} - \frac{1}{3}\delta_{jk}\delta_{mn}.$$

(Exercise: Work out the correspondence between the A^j_k and the 8 G_i.) Then in addition to the octet A^j_k, we can define another octet

$$D^j_k = \{A^j_i, A^i_k\} - \frac{2}{3}\delta^j_k(A^m_n A^n_m).$$

Then the mass formula reads

$$M = M_0 + M_1 \left\langle A^3_3 \right\rangle + M_2 \left\langle D^3_3 \right\rangle.$$

It is easy to show that $A_3^3 \propto Y$ and $D_3^3 \propto \vec{I}^2 - \frac{Y^2}{4}$ + constant. Thus we arrive at the Gell-Mann-Okubo formula, so that $\Delta M = aY + b\,[I(I+1) - \frac{1}{4}Y^2]$.

For mesons, $a = 0$ by C invariance.

For triangular representation, $I = 1 + \frac{1}{2}Y$, so that $\Delta M \propto Y$.

For further discussion of mass splitting, see the book by Carruthers and references therein.

CHAPTER III: INTRODUCTION TO SU_4

SU_4 - Fundamental Representation

SU_4 is defined as that special group of unitary 4 by 4 matrices with determinant +1. The new degree of freedom will be associated with the charmed quark. Before writing down the fundamental representation, a slight variation of notation will be explained. Instead of using indices 1 through 4 to label the 4 degrees of freedom, the charmed quark will be assigned the index 0. This is for convenience later on when the up quark and charmed quark will be grouped together as a doublet of objects with the same charge.

The fundamental representation, $\underline{4}$, of SU_4 will be denoted by

$$\psi \equiv \begin{pmatrix} \psi^0 \\ \psi^1 \\ \psi^2 \\ \psi^3 \end{pmatrix} \equiv \begin{pmatrix} c \\ u \\ d \\ s \end{pmatrix} .$$

We proceed in the by-now familiar way. An infinitesimal transformation is of the form

$$U(\vec{\epsilon}) = 1 + i\vec{\epsilon} \cdot \vec{G}.$$

The generators, $\vec{G}$, are still hermitian and traceless, but now, of course, they are 4 by 4 matrices. There are 15 such matrices, only 3 of which can be simultaneously diagonal.

Define $G_i = \frac{1}{2}\lambda_i$. In order to retain the isospin and unitarity spin sub-groups, the first 8 generators are chosen identical to those of SU_3. Just as for SU_3, the remaining nondiagonal matrices, which are necessary to allow transformations mixing charm with the other 3 kinds of quarks, are chosen in imitation of the nondiagonal Pauli matrices. The last, diagonal, generator is chosen in such a way that it is a multiple of the identity, so far as SU_3 is concerned, while distinguishing charmed and uncharmed quarks.

$$\lambda_i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \lambda_i & \\ 0 & & & \end{pmatrix} \qquad i = 1, 8$$

$$\lambda_9 = \left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & \bigcirc \\ \hline \multicolumn{2}{c|}{\bigcirc} & \bigcirc \end{array}\right) \qquad \lambda_{10} = \left(\begin{array}{cc|c} 0 & -i & \\ i & 0 & \bigcirc \\ \hline \multicolumn{2}{c|}{\bigcirc} & \bigcirc \end{array}\right)$$

$$\lambda_{11} = \left(\begin{array}{c|cc} & 1 & 0 \\ \bigcirc & 0 & 0 \\ \hline \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} & \multicolumn{2}{c}{\bigcirc} \end{array}\right) \qquad \lambda_{12} = \left(\begin{array}{c|cc} & -i & 0 \\ \bigcirc & 0 & 0 \\ \hline \begin{array}{cc} i & 0 \\ 0 & 0 \end{array} & \multicolumn{2}{c}{\bigcirc} \end{array}\right)$$

$$\lambda_{13} = \left(\begin{array}{c|cc} & 0 & 1 \\ \bigcirc & 0 & 0 \\ \hline \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} & \multicolumn{2}{c}{\bigcirc} \end{array}\right) \qquad \lambda_{14} = \left(\begin{array}{c|cc} & 0 & -i \\ \bigcirc & 0 & 0 \\ \hline \begin{array}{cc} 0 & 0 \\ i & 0 \end{array} & \multicolumn{2}{c}{\bigcirc} \end{array}\right)$$

$$\lambda_{15} = \frac{1}{\sqrt{6}} \left(\begin{array}{cc|c} -3 & 0 & \\ 0 & 1 & \bigcirc \\ \hline \multicolumn{2}{c|}{\bigcirc} & \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \end{array}\right)$$

In my numbering I follow the convention used in D. Amati, H. Bacry, J. Nuyts, and J. Prentki, Nuovo Cimento 34, 1732 (1964). In this paper you will find many useful mathematical properties of SU_4 presented in a form convenient for reference. In an appendix to this chapter, I reproduce the most useful tables.

These generators satisfy the following algebra

$$\left[\frac{\lambda_i}{2}, \frac{\lambda_j}{2}\right] = i\, f_{ijk} \frac{\lambda_k}{2},$$

where the f_{ijk} are simple constants that can be worked out if needed.

Charm can be defined as the eigenvalues of

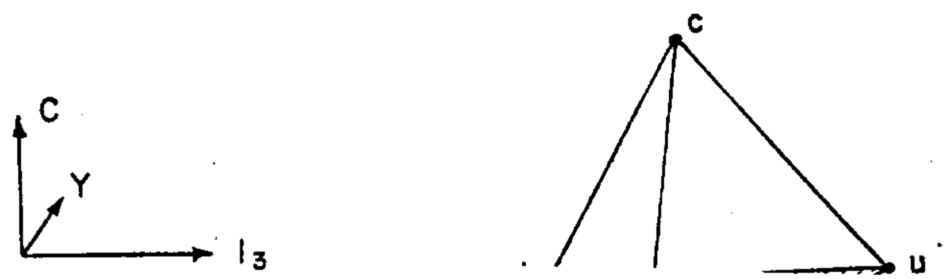
$$C = \frac{1}{4}(1 - \sqrt{6}\ \lambda_{15}) = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}.$$

So charm is not a generator of SU_4.* With this definition of charm, the charge operator is

$$Q = I_3 + \frac{Y}{2} + \frac{2}{3} C. \qquad \begin{pmatrix} \text{to make charge of} \\ \text{c equal charge of u.} \end{pmatrix}$$

Charge is not a generator of SU_4 either. It is possible to make other charge assignments, but we shall follow the lepton-hadron analogy of Bjorken-Glashow-Iliopoulous-Maiani.

The fundamental representation may be pictured as a tetrahedron:



The axes for the horizontal plane are our old friends I_3 and Y. The vertical axis is charm C. The four faces of the tetrahedron obviously form 4 SU_3 subgroups. The most familiar is, of course, the one formed from (u,d,s). Under this SU_3, the charmed quark c is a singlet.

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*The inclusion of the unit matrix 1 enlarges the group to U_4.

Conjugate Representation

The conjugate representation, $\underset{\sim}{4}^*$, is defined in the usual way with generators $-G^*$. As in SU_3, since the eigenvalues of $-G^*$ are different from those of $+G$, these turn out to be independent representations.

Products of Representations

As usual, only three invariant tensors exist in SU_4. These are the completely antisymmetric symbols and the Kronecker delta symbol.

$$\epsilon_{\alpha\beta\rho\sigma} = \epsilon^{\alpha\beta\rho\sigma} \quad \text{and} \quad \delta_\alpha^{\ \beta} \qquad \alpha, \beta, \rho, \sigma = 0, 1, 2, 3.$$

Now consider the product $\underset{\sim}{4} \otimes \underset{\sim}{4}$. One representation is $\psi^\alpha\psi^\beta$. The irreducible representations are found by picking out the permutation symmetries of the indices.

$$\psi^\alpha\psi^\beta = \underbrace{\frac{1}{2}(\psi^\alpha\psi^\beta + \psi^\beta\psi^\alpha)}_{\underset{\sim}{10}} + \underbrace{\frac{1}{2}(\psi^\alpha\psi^\beta - \psi^\beta\psi^\alpha)}_{\underset{\sim}{6}} .$$

These should not be confused with SU_3 representations of the same dimension.

Similarly the product $\underset{\sim}{4}^* \otimes \underset{\sim}{4}^*$ can be reduced as follows

$$\psi_\alpha\psi_\beta = \underbrace{\frac{1}{2}(\psi_\alpha\psi_\beta + \psi_\beta\psi_\alpha)}_{\underset{\sim}{10}^*} + \underbrace{\frac{1}{2}(\psi_\alpha\psi_\beta - \psi_\beta\psi_\alpha)}_{\underset{\sim}{6}^*}$$

$$(\psi_\alpha\psi_\beta - \psi_\beta\psi_\alpha) \quad \underset{\text{"transforms like"}}{\cong} \quad \epsilon_{\alpha\beta\rho\sigma}\psi^\rho\psi^\sigma$$

(This illustrates the utility of the tensor notation.) Consequently, $\underset{\sim}{6}$ and $\underset{\sim}{6}^*$ are equivalent representations.

At this point it is appropriate to point out an important way in which SU_4 differs from both SU_2 and SU_3. In both SU_2 and SU_3 the judicious application of the epsilon symbol can reduce any tensor with a pair of antisymmetric indices to a tensor of lower rank. This fact leads to the theorem that the most general irreducible representation in either SU_2 or SU_3 is a traceless tensor completely symmetric in all its upper and lower indices. In SU_4 the epsilon symbol can only exchange a pair of antisymmetric upper (lower) indices for a pair of antisymmetric lower (upper) indices. Therefore, the most general irreducible representation in SU_4 can possess a mixed symmetry.

For many purposes, this doesn't make the tensor notation any less useful. For example.

$$\psi_\alpha \psi^\alpha \text{ is a singlet}$$

$$\epsilon_{\alpha\beta\rho\sigma}\psi^\beta\psi^\rho\psi^\sigma \cong \psi_\alpha, \text{ and}$$

$$\epsilon_{\alpha\beta\rho\sigma}\psi^\alpha\psi^\beta\psi^\rho\psi^\sigma \text{ is a singlet.}$$

SU_3 Subgroups of Definite Charm

Identification of those SU_3 subgroups having unique charm eigenvalues parallels the reduction performed earlier on SU_3 into SU_2 subgroups. Although these are fancier ways, the simplest is to count the number of charmed quarks.

For example, the fundamental representation, $\underset{\sim}{4}$, ψ^α splits into an SU_3 triplet $\underset{\sim}{3}$ ψ^i $(i = 1, 2, 3)$ with charm zero and a singlet $\underset{\sim}{1}$ ψ^0 with charm $C = +1$. Similarly, we may decompose $(\underset{\sim}{10})^{\alpha\beta} = \{\psi^\alpha, \psi^\beta\}$ as follows

	SU_3 Representation	Charm C
None Zero $(\underset{\sim}{10})^{ij}$	$\underset{\sim}{6}$	0
One Zero $(\underset{\sim}{10})^{0i}$	$\underset{\sim}{3}$	+1
Two Zero $(\underset{\sim}{10})^{00}$	$\underset{\sim}{1}$	+2

Similarly, for $(\underset{\sim}{6})^{\alpha\beta} = [\psi^\alpha, \psi^\beta]$:

None Zero $(\underset{\sim}{6})^{ij}$	$\underset{\sim}{3}^*$	0
One Zero $(\underset{\sim}{6})^{0i}$	$\underset{\sim}{3}$	+1

Regular Representation $\underset{\sim}{15}$

In the by-now-familiar way, the regular representation is defined from the structure constants

$$(F_k)_{ij} = -if_{ijk} \qquad i,j,k = 1,2,\ldots,15.$$

As usual, one can establish the isomorphism

$$F_k \cong \tfrac{1}{2}\,\overline{\psi}\,\lambda_k\psi \equiv \tfrac{1}{2}\,\psi_\alpha(\lambda_k)_{\alpha\beta}\psi^\beta .$$

SU_4 Mesons

Mesons correspond as usual to quark-antiquark pairs. Consider $\underset{\sim}{4} \otimes \underset{\sim}{4}^*$. The reduction is essentially identical to the reduction of $\underset{\sim}{3} \otimes \underset{\sim}{3}^*$ in SU_3.

$$\underset{\sim}{4} \otimes \underset{\sim}{4}^* = \psi^\alpha\psi_\beta = \underbrace{[\psi^\alpha\psi_\beta - \tfrac{1}{4}\delta_\beta^{\ \alpha}]}_{(\underset{\sim}{15})_\beta^{\ \alpha}} + \underbrace{\tfrac{1}{4}\delta_\beta^{\ \alpha}}_{\underset{\sim}{1}} .$$

As indicated previously, these $\underset{\sim}{15}$ transform like the regular representation. Now write out the states.

$(15)_3^1 = u\bar{s}$	K^+		$(15)_0^1 = u\bar{c}$	$\overline{D^o}$
$(15)_3^2 = d\bar{s}$	K^o		$(15)_0^2 = d\bar{c}$	D^-
$(15)_2^1 = u\bar{d}$	π^+		$(15)_0^3 = s\bar{c}$	F^-
$(15)_1^2 = d\bar{u}$	π^-		$(15)_1^0 = c\bar{u}$	D^o
$(15)_1^3 = s\bar{u}$	K^-		$(15)_2^0 = c\bar{d}$	D^+
$(15)_2^3 = s\bar{d}$	$\overline{K^o}$		$(15)_3^0 = c\bar{s}$	F^+

$$\left.\begin{aligned}
(15)_1^1 &= \frac{3}{4}u\bar{u} - \frac{1}{4}(d\bar{d} + s\bar{s} + c\bar{c}) \\
(15)_2^2 &= \frac{3}{4}d\bar{d} - \frac{1}{4}(u\bar{u} + s\bar{s} + c\bar{c}) \\
(15)_3^3 &= \frac{3}{4}s\bar{s} - \frac{1}{4}(u\bar{u} + d\bar{d} + c\bar{c}) \\
(15)_0^0 &= \frac{3}{4}c\bar{c} - \frac{1}{4}(u\bar{u} + d\bar{d} + s\bar{s})
\end{aligned}\right\} \text{not all independent since the trace (their sum) must be zero.}$$

The physical combinations can only be identified by appealing to SU_2 and SU_3. The π^o and η have already been identified as

$$\pi^o = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) \propto (15)_1^1 - (15)_2^2$$

$$\eta = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s}) \propto (15)_1^1 + (15)_2^2 - (15)_3^3 .$$

The remaining state is now fixed to be orthogonal to these two and to the singlet

$$\eta_{15} = \frac{1}{\sqrt{12}}(u\bar{u} + d\bar{d} + s\bar{s} - 3c\bar{c}) \propto (15)_0^0 .$$

In addition to these 15 states, there is the SU_4 singlet $\underset{\sim}{1}$

$$\eta_1 = \frac{1}{2}(u\bar{u} + d\bar{d} + s\bar{s} + c\bar{c}).$$

Except for the π^0, required to have the form above by isospin symmetry, ?
the mixtures of η, η_{15}, and η_1 which correspond to physical states depends in detail on the way SU_4 is broken. We might still imagine that it is broken in a way which approximately preserves SU_3, which would suggest that η is as given above, and the remaining two states are apt to be

$$\eta' = X^0 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s})$$

$$\eta_c = c\bar{c}.$$

One can essentially read off the SU_3 subgroups from our list of particles, but it is useful to proceed systematically to identify them as we did before for $\underset{\sim}{4}$, $\underset{\sim}{6}$, and $\underset{\sim}{10}$.

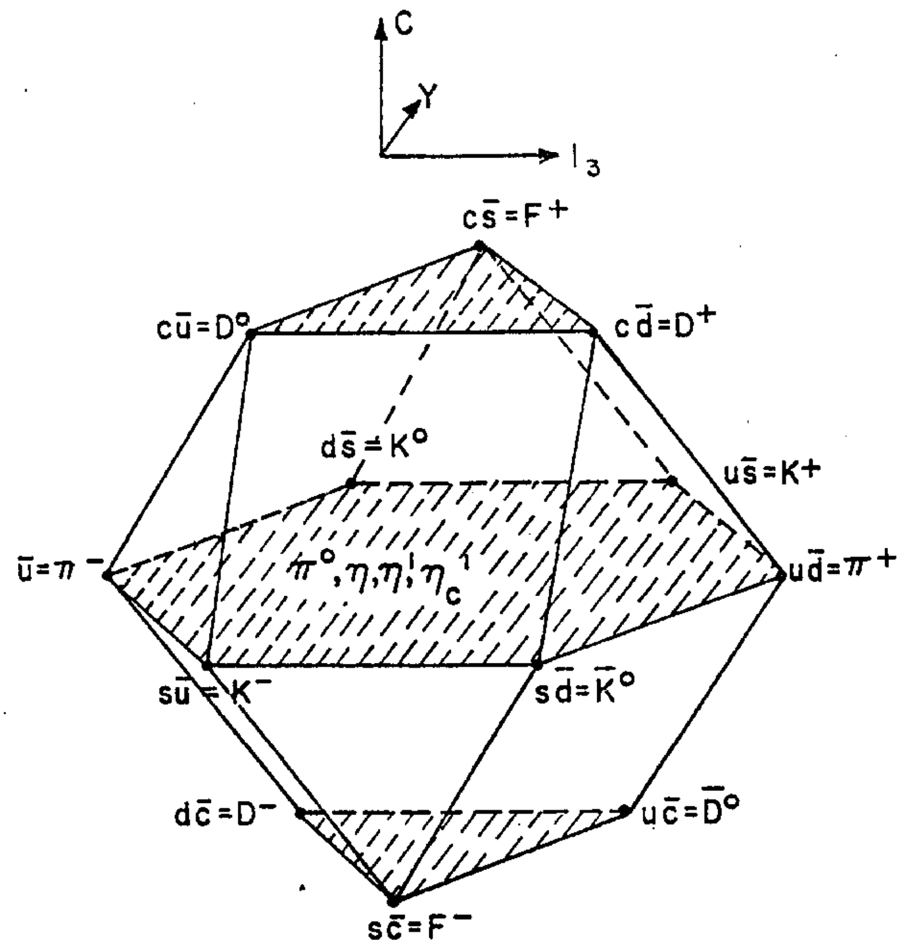
		SU_3 Content	C
None Zero	$(\underset{\sim}{15})_j^i$	$\underset{\sim}{8} + \underset{\sim}{1}$	0
One Zero	$(\underset{\sim}{15})_0^i$	$\underset{\sim}{3}$	+1
	$(\underset{\sim}{15})_i^0$	$\underset{\sim}{3}^*$	-1
Two Zero	$(\underset{\sim}{15})_0^0$	$\underset{\sim}{1}$	0

We appear to have identified 16 states but the SU_3 singlet formed from "none zero" is not independent of the singlet with two zeros since $(15)_\alpha^\alpha = 0$. To see this explicitly, observe that

$$(15)_j^i = \underbrace{\psi^i\psi^j - \frac{1}{3}\delta_j^i(\psi^k\psi_k)}_{(\underset{\sim}{8})_j^i} - \frac{1}{3}\delta_j^i(\underset{\sim}{15})_0^0 .$$

(Exercise: Verify this.)

Of course, the SU_4 singlet state remains a singlet under SU_3. The 16 states can be displayed on a three dimensional weight diagram:



This figure, as well as the names for the charmed mesons, has been taken from a paper by M. K. Gaillard, B. W. Lee and J. Rosner, "Search for Charm," FERMILAB-Pub-74/86-THY, August 1974, and "Addendum," FERMILAB-Pub-75/14-THY, January, 1975 (both to be published in Rev. Mod. Phys., April 1975). The first of the two preprints was written shortly before the discovery of the resonances at 3.1 and 3.7 GeV. The basic phenomenology of charmed particles is covered thoroughly and this is as good a time as any to recommend it to you. In many ways, these lectures are but a prelude to their discussion. Hereafter, we shall refer to them as GLR. For more about η_c, see B. W. Lee and C. Quigg, FERMILAB-74/110-THY, December 1974.

Of course, we can anticipate SU_4 multiplets of mesons at higher mass. The identification of the above 16 states with the pseudoscalar mesons assumes that the total angular momentum of the quark-antiquark system is zero. In the nonrelativistic quark model, this is identified with no orbital excitation ($L = 0$) and a coupling of spins to zero ($S = 0$). If, instead, the spins couple to one, we obtain a vector meson multiplet. Consequently, in addition to the usual nonet consisting of the K^*, ρ, ω, ϕ, we add seven other particles called naturally, D^*, F^*, and $\phi_c = c\bar{c}$. It is this last state which has been identified with the lowest narrow resonance at 3.1 GeV discovered at SPEAR (where it was called ψ) and at BNL (where it was called J). This gives us a rough idea how much heavier the charmed quark must be. For further discussion of the consequences of this interpretation, see GLR (especially the Addendum) and any recent issue of Phys. Rev. Letters.

SU_4 Baryons

The baryons transform as 3 quarks so we must consider $\underset{\sim}{4} \otimes \underset{\sim}{4} \otimes \underset{\sim}{4}$. There are 64 independent states here, all with the general structure $\psi^\alpha\psi^\beta\psi^\gamma$. Part of the reduction has already been discussed.

$$\underset{\sim}{4} \otimes \underset{\sim}{4} \otimes \underset{\sim}{4} = (\underset{\sim}{10} \oplus \underset{\sim}{6}) \otimes \underset{\sim}{4} = \underset{\sim}{10} \otimes \underset{\sim}{4} \oplus \underset{\sim}{6} \otimes \underset{\sim}{4}.$$

Consider the 40 states in $\underset{\sim}{10} \times \underset{\sim}{4}$ first. The $\underset{\sim}{10}$ was shown to be symmetric in its two indices.

$$(\underset{\sim}{10})^{\alpha\beta} = \psi^\alpha\psi^\beta + \psi^\beta\psi^\alpha = \{\psi^\alpha, \psi^\beta\} .$$

The representations contained in $\underset{\sim}{10} \otimes \underset{\sim}{4}$ must respect this symmetry. One irreducible representation is easy to write down--the completely symmetric one.

$$(\underset{\sim}{20})^{\alpha\beta\gamma} \equiv \psi^\alpha\psi^\beta\psi^\gamma + \psi^\beta\psi^\gamma\psi^\alpha + \psi^\gamma\psi^\alpha\psi^\beta + \psi^\beta\psi^\alpha\psi^\gamma + \psi^\alpha\psi^\gamma\psi^\beta + \psi^\gamma\psi^\beta\psi^\alpha .$$

As indicated, this representation has dimension 20. (Exercise: show this.)

A second tensor can be formed using the epsilon symbol

$$(\underset{\sim}{20}'_S)_{\rho\sigma}^{\ \ \alpha} \equiv \epsilon_{\rho\sigma\beta\gamma} \{\psi^\alpha, \psi^\beta\} \psi^\gamma .$$

This expression is traceless as it stands. It forms a 20 dimensional representation of mixed symmetry. (The subscript S reminds us that it is symmetric in the first two quarks.)

The only other tensor which might be considered,

$$\epsilon_{\rho\alpha\beta\gamma}\{\psi^\alpha, \psi^\beta\} \ \psi^\gamma ,$$

is obviously identically zero. This is nice because the 40 states have already been accounted for

$$\underset{\sim}{10} \otimes \underset{\sim}{4} = \underset{\sim}{20} + \underset{\sim}{20}'_S .$$

Now consider $\underset{\sim}{6} \otimes \underset{\sim}{4}$. Recall that $\underset{\sim}{6}$ is antisymmetric in its indices

$$(\underset{\sim}{6})^{\alpha\beta} \equiv \psi^\alpha\psi^\beta - \psi^\beta\psi^\alpha = [\psi^\alpha, \psi^\beta].$$

We consider the decomposition of $[\psi^\alpha, \psi^\beta]\psi^\gamma$. One possibility is the totally antisymmetric combination

$$(\underset{\sim}{4}^*)_\rho \equiv \epsilon_{\alpha\beta\gamma\rho}[\psi^\alpha, \psi^\beta]\psi^\gamma .$$

As indicated, it obviously transforms as $\underset{\sim}{4}^*$.

Once more there is only one other tensor that can be constructed

$$(\underset{\sim}{20}'_A)_{\rho\sigma}^{\ \alpha} \equiv \epsilon_{\rho\sigma\beta\gamma} [\psi^\alpha, \psi^\beta] \psi^\gamma \ - (\text{traces}).$$

This exhausts the 24 states of $\underset{\sim}{6} \otimes \underset{\sim}{4}$. (Exercise: Work out the (traces) terms.)

In summary,

$$\underset{\sim}{4} \otimes \underset{\sim}{4} \otimes \underset{\sim}{4} = \underset{\sim}{20} + \underset{\sim}{20}'_S + \underset{\sim}{20}'_A + \underset{\sim}{4}^*.$$

SU_3 Subgroups of Baryon Multiplets

The simplest way to visualize the SU_4 baryons involves detailing the SU_3 subgroups. Consider the totally symmetric $\underset{\sim}{20}$:

$$(\underset{\sim}{20})^{\alpha\beta\gamma}.$$

Proceeding as we did with $\underset{\sim}{4}$, $\underset{\sim}{6}$, $\underset{\sim}{10}$. and $\underset{\sim}{15}$, we make a little table:

		SU_3 Content	C
None Zero	$(\underset{\sim}{20})^{ijk}$	$\underset{\sim}{10}$	0
One Zero	$(\underset{\sim}{20})^{0ij}$	$\underset{\sim}{6}$	+1
Two Zero	$(\underset{\sim}{20})^{00i}$	$\underset{\sim}{3}$	+2
Three Zero	$(\underset{\sim}{20})^{000}$	$\underset{\sim}{1}$	+3

(Check: 10 + 6 + 3 + 1 = 20.) We recognize the C = 0 subgroup as the familiar $(3/2)^+$ decuplet.

The SU_3 baryon octet must be contained in the two $\underset{\sim}{20}$'s of mixed symmetry. As before, where Fermi statistics is imposed on the total wavefunction, the 40 states which result are not all allowed. Only twenty states can be formed (for each value of angular momentum) consistent with the Pauli principle.

In any case, the 20 allowed states will be built from $\underset{\sim}{20}_S'$ and $\underset{\sim}{20}_A'$ combined with spin in such a way to form a symmetric product of spin $\times$ SU_4. So let us analyze the SU_3 content of $(\underset{\sim}{20}')_{\rho\sigma}^{\alpha} \cong \epsilon_{\rho\sigma\beta\gamma}\psi^{\alpha}\psi^{\beta}\psi^{\gamma}$ - (traces). We proceed as before to set various indices to zero.

Consider $(\underset{\sim}{20}')_{ij}^{0}$ $(i,j = 1,2,3)$. Because we have required ρ and σ nonzero, either β or γ must be zero. Since $\alpha = 0$, these states contain two charmed quarks and one uncharmed quark. Multiplying by ϵ^{kij}, we recall that in SU_3 states antisymmetric in two lower indices transform like a tensor with a single upper index. Thus, these form an SU_3 triplet $\underset{\sim}{3}$ of charm +2.

$$(20')_{ij}^{0}: \qquad \begin{matrix} c[c,d] & & c[c,u] \\ & c[c,s] & \end{matrix} \equiv \begin{matrix} X_d^{+} & & X_u^{++} \\ & X_s^{+} & \end{matrix},$$

where we have indicated the names given in GLR. We see an isodoublet and isosinglet here.

Next consider

$$(\underset{\sim}{20}')_{0i}^{0} = \psi^{0}\epsilon_{ijk}\psi^{j}\psi^{k}.$$

This obviously transforms as 3^* and, containing a single charmed quark, has $C = +1$.

$$(\underset{\sim}{20}')_{0i}^{0}: \qquad \begin{matrix} & c[u,d] & \\ c[d,s] & & c[u,s] \end{matrix} \equiv \begin{matrix} & C_o^{+} & \\ A^{o} & & A^{+} \end{matrix}.$$

Again, we see a non-strange isosinglet C_o^+ and an isodoublet A^+, A^o with $Y = 0$.

Another subgroup with $C = +1$ is formed from $(\underset{\sim}{20}')_{ij}^{k}$. With 3 upper indices and 3 independent lower indices, we appear to have 9 states. However, the trace condition on $\underset{\sim}{20}'$ reads

$$(\underset{\sim}{20'})_{ij}^{\;i} + (\underset{\sim}{20'})_{0j}^{\;0} = 0\,,$$

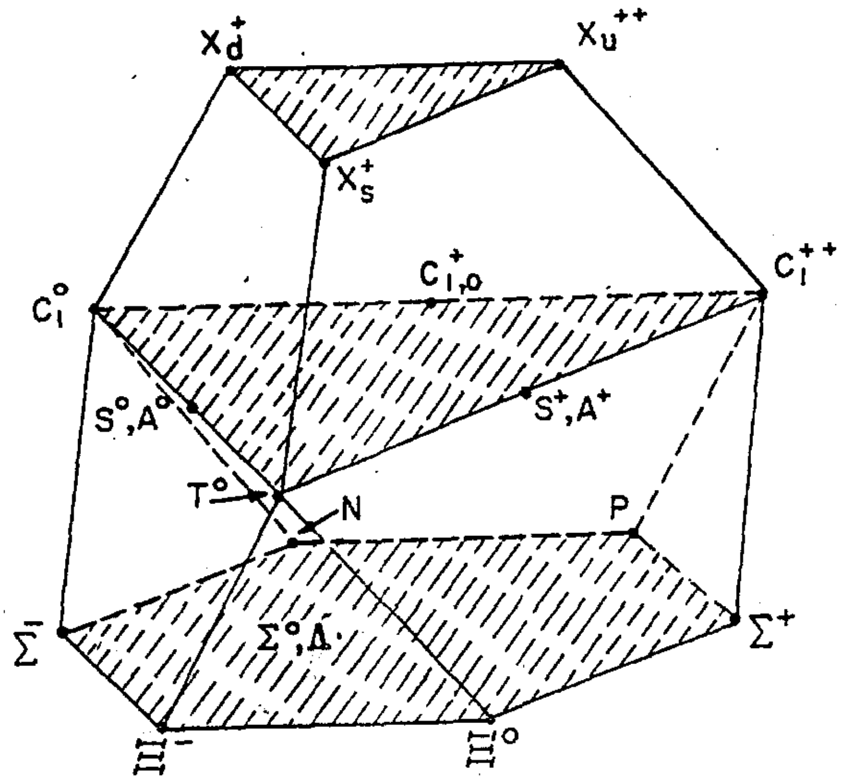
so only six of the nine states are new, the remaining ones forming 3^*. They are

$$\begin{array}{ccccccc} cdd & c\{u,d\} & cuu & & C_1^{\;o} & C_1^{\;+} & C_1^{\;++} \\ & c\{s,d\} & c\{s,u\} & \equiv & S^o & S^+ & \\ & & css & & & T^o & \end{array}$$

Those states whose quark content is the same as those in 3^* above have been chosen orthogonally. Thus we choose symmetric combinations of $\{u,d\}$, $\{s,d\}$, and $\{s,u\}$. Obviously, we obtain a nonstrange isovector C_1, an isodoublet S with Y = 0, and an isosinglet T^o with Y = -1.

So far, we have identified $\underset{\sim}{3} + \underset{\sim}{3}^* + \underset{\sim}{6} = 12$ states. It will come as no surprise that the remaining eight states with C = 0 are the familiar baryon octet. They come from $(\underset{\sim}{20'})_{0j}^{\;k}$. [There appear to be nine states here, but, because $(\underset{\sim}{20'})_{0k}^{\;k} = 0$, only eight are independent.]

These states may also be represented on a three-dimensional weight diagram, which we take from GLR:



In the product of $\underset{\sim}{4} \otimes \underset{\sim}{4} \otimes \underset{\sim}{4}$, there also occurs a set of four states transforming as $\underset{\sim}{4}^*$. This representation is totally antisymmetric in the 3 quarks. To realize a physical baryon, we must combine this with a totally antisymmetric spin state so that the wave function would be symmetric in $SU_4 \times$ spin. However, there doesn't exist a totally antisymmetric spin state formed from three spin $\frac{1}{2}$ particles, so, once again, the Pauli principle excludes $\underset{\sim}{4}^*$ as a candidate for physical baryons.

For those interested in mass formulas, I recommend GLR for a discussion on what we can expect from first-order SU_4 breaking. It is questionable how valid a first-order perturbative calculation will be since SU_4 is so much more badly broken than is SU_3. However, until we obtain experimental input, we can do no better. Although our expectations for the masses of the charmed baryons and mesons may turn out to be incorrect, the existence of states with the quantum numbers given above is the crucial test of the whole scheme. Some simple conclusions can be drawn from the SPEAR data which fixes the mass range of the lightest charmed mesons, presumably the $D^{\pm}$, D^{o}, and $\overline{D^{o}}$. In addition to the resonance at 3.1 GeV, another very narrow resonance has been observed at 3.7 GeV, which is interpreted in this charm scheme as a "radial" excitation of the c-$\bar{c}$ pair. Because it is so narrow, its decay into say D^+D^- must be kinematically forbidden. Consequently, we expect $m_D > 1.85$ GeV. However, another bump in e^-e^+ annihilation has been observed at 4.15 GeV, which may be a second radial excitation. Regardless, it is quite broad suggesting that it may be above the threshold for decay into pairs of charmed mesons. Consequently, we would expect

$m_D < 2.07$ GeV. Thus, the D meson probably lies in the range (1.85, 2.07). As we shall see, the lightest charmed particles are expected to decay weakly, indeed, this will be the true signature of a new quantum number which is conserved by strong and electromagnetic interactions. We shall discuss later into which channels they will decay.

APPENDIX TO CHAPTER III

Appendix from D. Amati, H. Bacry, J. Nuyts and J. Prentki, Nuovo Cimento 34, 1732 (1965).

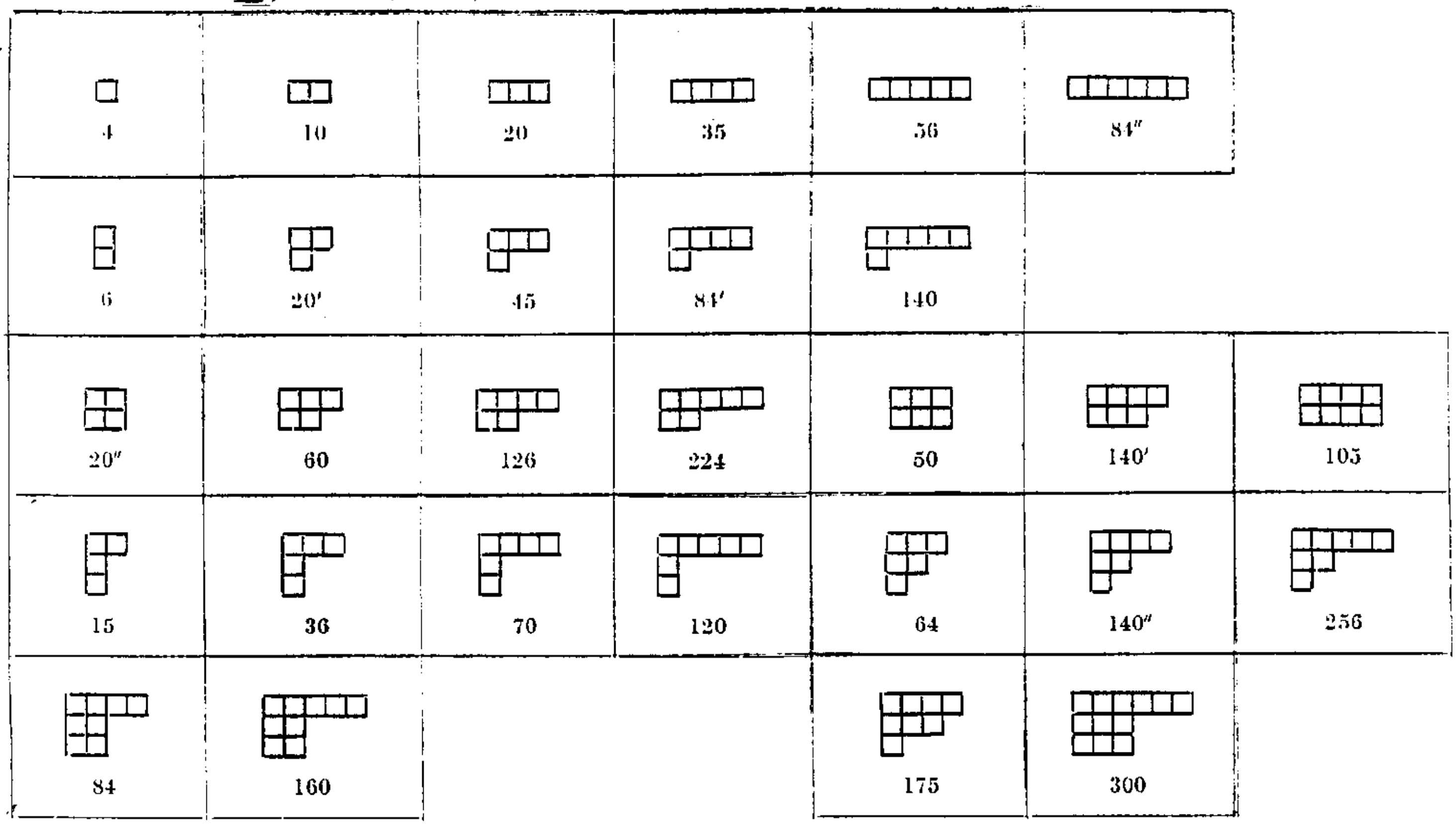


TABLE I. – *Product decomposition of representations in SU_4.*

4	$\bar{4}$	6	10	$\overline{10}$	15	20	$\overline{20}$	$20'$	$\overline{20}'$	$20''$	
$6+10$	$1+15$	$\bar{4}+20'$	$20+20'$	$\bar{4}+\overline{36}$	$4+\overline{20}'+36$	$35+45$	$\overline{10}+70$	$15+20''+45$	$6+\overline{10}+64$	$\overline{20}'+60$	4
		$1+15+20''$	$15+45$	$\overline{15}+\overline{45}$	$6+10+\overline{10}+64$	$36+84'$	$\overline{36}+\overline{84}'$	$4+\overline{20}'+36+60$	$\bar{4}+20'+\overline{36}+\overline{60}$	$6+50+64$	6
			$20''+35+45$	$1+15+84$	$6+10+64+70$	$56+60+84'$	$\bar{4}+\overline{36}+\overline{160}$	$\overline{20}'+36+60+84'$	$\bar{4}+20'+\overline{36}+140''$	$\overline{10}+64+126$	10
					$1+15+15+20''+45+\overline{45}+84$	$20+20'+\overline{120}+\overline{140}''$	$\overline{20}+\overline{20}'+\overline{120}+\overline{140}''$	$\bar{4}+20+20'+\overline{20}'+\overline{36}+\overline{60}+140''$	$4+\overline{20}+\overline{20}'+\overline{20}'+36+60+140''$	$15+20''+45+\overline{45}+175$	15
						$50+84''+126+140$	$1+15+84+300$	$64+70+126+140$	$15+45+84+256$	$\overline{36}+140''+224$	20
								$6+10+\overline{10}+50+64+64+70+126$	$1+15+15+20''+\overline{45}+45+84+175$	$\bar{4}+20'+\overline{36}+\overline{60}+140'+140''$	20′
										$1+15+20''+84+105+175$	20″

IV. SU_4 STRUCTURE OF ELECTROMAGNETIC AND WEAK CURRENTS

Now we shall turn to a discussion of how the charmed quark enters the electromagnetic and weak currents. First, we shall briefly indicate what the currents are expected to look like and what the weak decays of the lightest charmed particles will be. Later, we shall discuss the implications of this scheme for deep inelastic lepton scattering.

Structure of the Electromagnetic Current

The current is

$$J_\mu = \sum_\alpha e_\alpha \overline{\psi^\alpha} \gamma_\mu \psi^\alpha ,$$

where e_α is the charge of the corresponding quark field. Ignoring the space-time structure, this takes the form in our scheme

$$J = \frac{2}{3}(\bar{c}c + \bar{u}u) - \frac{1}{3}(\bar{d}d + \bar{s}s),$$

where we have suppressed the electronic charge e.

Just as in SU_3, G_U parity prohibited the transition

$$\gamma \nrightarrow K^o\overline{K^o} ,$$

so, in SU_4, an SU_2 subgroup $P = \begin{pmatrix} c \\ u \end{pmatrix}$ and a G_P-parity operator can be defined to suppress the transition

$$\gamma \nrightarrow D^o\overline{D^o} .$$

How well the selection rule on $K^o\overline{K^o}$ works is not known. I suspect that SU_4 is so badly broken that the $D^o\overline{D^o}$ channel may not be suppressed at all, certainly not until energies large compared to threshold.

Structure of the Weak Current

The purely leptonic part of the (charged) weak current is of the form

$$J_{\text{leptonic}}^{\text{weak}} = (\bar{\nu}_e \; \bar{\nu}_\mu)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^- \\ \mu^- \end{pmatrix} = \bar{\nu}_e e^- + \bar{\nu}_\mu \mu^- .$$

[The very important factors of $\gamma_\mu(1 - \gamma_5)$ associated with the space-time structure of the current have been suppressed here.] We have two doublets of particles with the same charge connected via the unit matrix. The unit matrix summarizes the experimental observations of μ-e universality.

The hadrons also participate in the weak interaction. The proposal here is that the hadronic part of the weak current can be written in a very similar way if quarks with the same charge are paired in doublets.

$$J_{\text{hadronic}}^{\text{weak}} = (\bar{c}\;\bar{u})\; U\begin{pmatrix} d \\ s \end{pmatrix}.$$

If SU_4 symmetry were exact, we would wish to be able to choose a new basis $\psi' = V\psi$ (for some unitary V) so that in the transformed basis, the hadronic current would be identical to the lepton current, i.e., in 4×4 language,

$$V\left(\begin{array}{c|c} 0 & U \\ \hline 0 & 0 \end{array}\right)V^+ = \left(\begin{array}{c|c} 0 & 1 \\ \hline 0 & 0 \end{array}\right).$$

This implies that U is a unitary 2×2 matrix. Now imagine that we were dealing with a field theory in which the only SU_4 breaking of strong interactions were due to the different quark masses carried by the strange and charmed quarks. Then one cannot rotate the basis (except within the isospin subgroup) without destroying the diagonality of the mass matrix.

However, one can still alter the phases of the fields and this, we will now show, is sufficient to reduce U to dependence on a single parameter.

To see this, consider that the most general 2×2 unitary matrix can be written as

$$U = e^{i\Delta} \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix},$$

where Δ is some real phase and the complex numbers α and β satisfy $|\alpha|^2 + |\beta|^2 = 1$. Suppose we define new (primed) fields, differing from the original ones by a phase

$$\begin{pmatrix} c \\ u \end{pmatrix} = e^{i\Gamma} \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \begin{pmatrix} c \\ u \end{pmatrix}' \qquad \begin{pmatrix} d \\ s \end{pmatrix} = e^{i\Lambda} \begin{pmatrix} e^{i\psi} & 0 \\ 0 & e^{-i\psi} \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix}'.$$

In terms of these new fields, the current assumes the form

$$J = e^{i(\Delta + \Lambda - \Gamma)} \, \widetilde{\overline{c}\;\overline{u}}' \begin{pmatrix} \alpha' & \beta' \\ -\beta'^* & \alpha'^* \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix}',$$

where $\alpha' = \alpha e^{i(\phi - \psi)}$ and $\beta' = \beta e^{i(\phi + \psi)}$. Clearly, we now have the freedom to remove all relative phases and reduce J to a function of a single parameter. To bring this expression above into conventional form, choose ϕ and ψ to make α' and β' purely imaginary and then choose $\Lambda - \Gamma$ to make the overall matrix real. Since $|\alpha|^2 + |\beta|^2 = 1$, we may define $\cos\theta_C = |\beta|$, $\sin\theta_C = |\alpha|$. Then the hadronic current becomes

$$J_H = \widetilde{\overline{c}\;\overline{u}}' \begin{pmatrix} -\sin\theta_C & \cos\theta_C \\ \cos\theta_C & \sin\theta_C \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix}'.$$

The sole remaining parameter θ_C is called the Cabibbo angle. Hereafter, we will drop the primes and assume that the fields are defined so that J takes this simple form. The upshot of this exercise was that, without loss of generality, we may assume that U is a real, orthogonal matrix. (The preceding discussion follows the argument of Glashow, Iliopoulous, and Maiani, op. cit.) In this context, the Cabibbo angle arises rather naturally. Why it assumes the value observed ($\tan^2 \theta_C = 0.056$) is unexplained. Whether it can somehow be related to the direction and magnitude of SU_4 breaking remains an outstanding question.

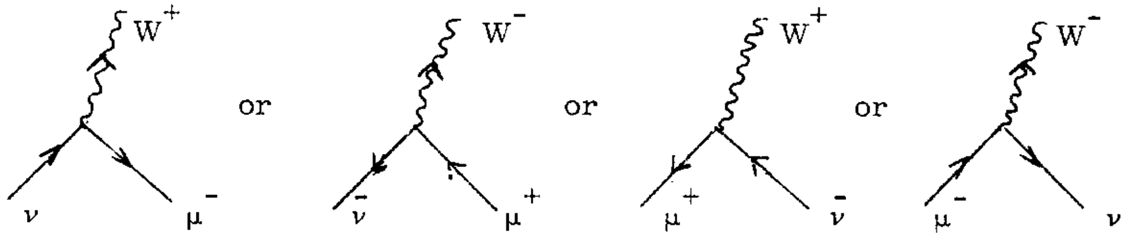
We assume the full weak current is the sum of the leptonic and hadronic pieces

$$J = J_L + J_H.$$

That we add these without further normalization constants expresses what we mean by lepton-hadron universality.

We imagine that this current couples in the Lagrangian to a charged weak boson, i.e., the interaction is of the form $J^\mu W_\mu^\dagger + W^\mu J_\mu^\dagger$. (For the discussion which follows, the W boson is inessential, although convenient, and we could equally well speak of an effective current-current interaction. In renormalizable gauge theories, however, the vector boson plays an essential role.)

Leptonic vertices:

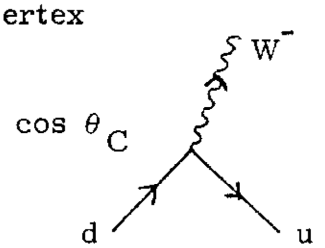


That is, we imagine we are dealing with a CP invariant, local theory so that all these vertices are allowed and equal. Henceforth, we shall draw only one, and you can infer the rest. We have similar diagrams for the electron with equal strength (μ-e universality).

Hadronic Vertices and Semi-Leptonic Decays

We will now draw the allowed vertices involving the quarks underlying the hadrons. Consider first

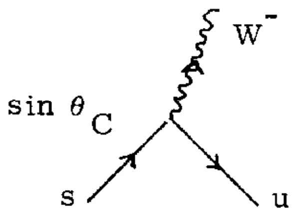
1. The u-d vertex



This vertex appears, for example, in β decay of the neutron, $n \to pe^-\bar{\nu}_e$, which results from reconversion of W^- into an electron-neutrino pair. Suppose, as with β decay, the d quark were a member of an initial bound state and, following W emission, the u quark is taken up by the final state, that is, we consider $W < f|\bar{u}\, d \cos\theta_c|i>$. How will the quantum numbers of the hadronic state have changed in going from $|i>$ to $|f>$? Clearly, the transition changes neither charm nor strangeness, $\Delta C = \Delta S = 0$. What about isospin? Since the initial quark and final quark both have $I = \frac{1}{2}$, when combined with the remaining quarks, the maximum change of isospin can be one. Consequently, only $\Delta I = 0,1$ are allowed. In summary, semileptonic decays of this type fulfill the selection rules $\Delta C = \Delta S = 0$, $\Delta I = 0,1$. Another important transition of this type is $\pi^- \to \mu^- \bar{\nu}_\mu$.

Were there no renormalizations due to strong interactions, the coupling for this vertex would be just cos θ_C times the leptonic vertex. (Current conservation implies that the conserved isovector current is not renormalized. However, the axial vector current is not conserved, and its coupling is changed by strong interactions.)

2. The s-u vertex

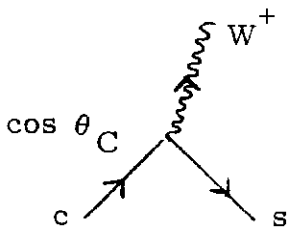


Semi-Leptonic Selection Rules

$\Delta C = 0$
$\Delta S = -1$
$\Delta I = \frac{1}{2}$
$\Delta Q = \Delta S$

This strangeness-changing vertex is responsible for decays such as $K^- \to \mu^- \bar{\nu}_\mu$. For semileptonic decays, generally, there is no change in charm; the strangeness increases by one as does the charge of the hadronic system (the famous $\Delta S = \Delta Q$ rule); the s quark is an isosinglet, while u is a member of an isodoublet, so isospin can change by $\frac{1}{2}$. If SU_3 were exact, then the ratio $\Gamma(K^- \to \mu^- \bar{\nu}_\mu)/\Gamma(\pi^- \to \mu^- \bar{\nu}_\mu)$ would be just $\tan^2 \theta_c \approx 0.06$.

3. The c-s vertex

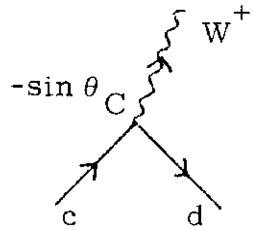


Semi-Leptonic Selection Rules

$\Delta C = 1$
$\Delta S = 1$
$\Delta I = 0$
$\Delta Q = \Delta S = \Delta C$

This vertex would be responsible for $F^+ \to \mu^+ \nu_\mu$, for example. In this theory, the Cabibbo-favored transition is from a charmed quark to a strange quark. Again we list the semi-leptonic selection rules.

4. The c-d vertex



Semi-Leptonic Selection Rules

$$\Delta C = 1$$
$$\Delta S = 0$$
$$\Delta I = \tfrac{1}{2}$$
$$\Delta Q = \Delta C$$

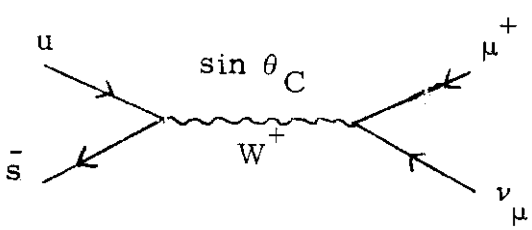
This vertex would be responsible for the decay $D^+ \rightarrow \mu^+ \nu_\mu$. This transition is Cabibbo-suppressed for charm-changing decays. Note once again the semi-leptonic selection rules.

Leptonic Decays

With only a few assumptions we can estimate the decay rates of the charmed hadrons. Assume that all matrix elements are SU_4 symmetric, but acknowledge the large SU_4 symmetry breaking by using the physical masses in the phase space calculations as usual. This will give us some ball park estimates of the rates.

Two-Body Decays

The K^+ is a $u\bar{s}$ system. Its two-body decay can be pictured



$\equiv T \sin\theta_C$.

This is the K_{2_μ} amplitude. The phase space factor for the decay into a muon and neutrino is just (see, e.g., Bjorken and Drell, vol. 1)

$$\text{phase space factor} = (\text{constant})\, M \left[1 - \left(\frac{m_\mu}{M}\right)^2\right],$$

where M is mass of the meson and m_μ is the muon mass. For mesons with the kaon mass or heavier, the factor in square brackets is within 5% of unity.

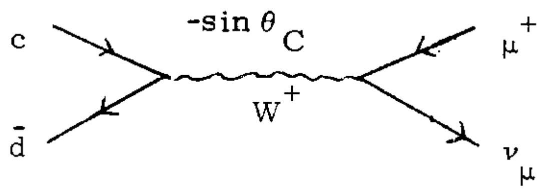
Therefore,

$$\text{phase space factor} = (\text{constants})\ M$$

will be used here. The partial width for the $K_{\mu 2}$ decay, then, is

$$\Gamma_{K_{2\mu}} \propto |T|^2 M_K \sin^2\theta_C.$$

The rates for the charmed mesons are now easy to write down. The D^+ is a $c\bar{d}$ system. Its two-body decay is



$$\Gamma_{D_{2\mu}} \propto |T|^2 M_D \sin^2\theta_C.$$

The decay of the F^+, a $c\bar{s}$ system, is not Cabibbo suppressed

$$\Gamma_{F_2} \propto |T|^2 M_F \cos^2\theta_C$$

In summary

$$\Gamma_{K_2} : \Gamma_{D_2} : \Gamma_{F_2} = 1 : \frac{M_D}{M_K} : \frac{M_F}{M_K}\cot^2\theta_C$$

Approximating $M_D = M_F = 2$ GeV, $M_K = \frac{1}{2}$ GeV, and $\cot^2\theta_C = 20$, results in

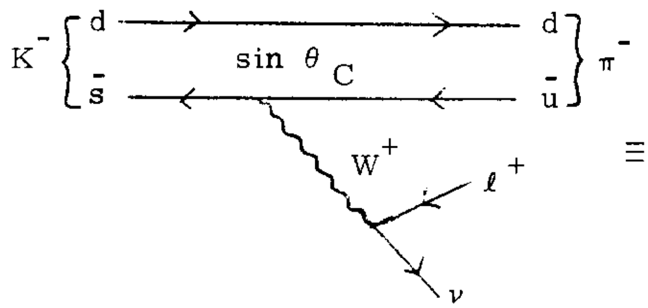
$$\Gamma_{K_2} : \Gamma_{D_2} : \Gamma_{F_2} = 1 : 4 : 80.$$

Since Γ_{K_2} is $0.5\times 10^8\ \text{sec}^{-1}$, one expects these two body, leptonic decays to have partial widths

$$\Gamma_{D_2} \sim 2\times 10^8\ \text{sec}^{-1} \text{ and } \Gamma_F \sim 4\times 10^9\ \text{sec}^{-1}.$$

Semi-Leptonic Three-Body Decays -

The diagrams for semi-leptonic decay into three-body final states feature one quark line which does not participate directly in the weak interaction. The K_3 decay, $K^o \to \pi^- \ell + \nu$, is represented by the diagram



$\equiv T \sin \theta_C.$

The phase space factor associated with a three-body final state is quite complicated in all its fine details. However, if the energy released in the decay is much larger than the masses of the final-state particles, one has the rule,

$$\text{phase space factor} \propto (\text{energy release})^5 .$$

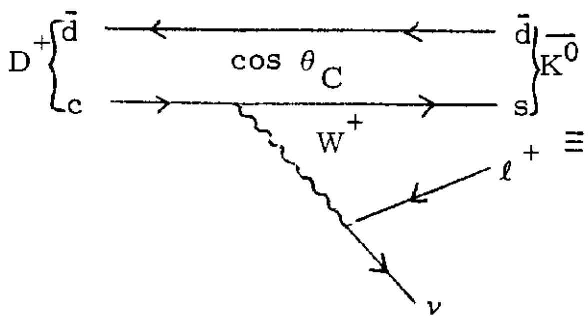
For heavy particles this is well approximated by the mass of the parent particle (see Bjorken and Drell, Vol. 1).

$$\text{phase space factor} \propto (\text{Mass})^5 .$$

The partial width for $K_{\ell 3}$ is of the form

$$\Gamma_{K_{\ell_3}} \propto M_K^5 \sin^2 \theta_C |T|^2 .$$

The decay of the D^+ can be treated in exactly the same way



$\equiv T \cos \theta_C.$

So

$$\Gamma_{D^+_3} \propto M_D^5 \cos^2 \theta_C |T|^2 .$$

The lack of Cabibbo suppression, together with the large increase in phase space, greatly enhances this decay mode

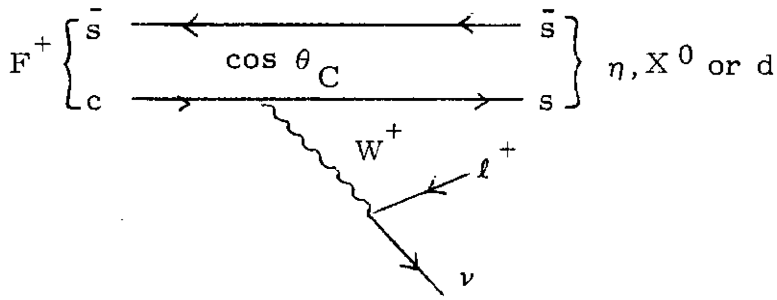
$$\frac{\Gamma_{D_3}}{\Gamma_{K_3}} = \left(\frac{M_D}{M_K}\right)^5 \cot^2 \theta_c \cong (4)^5 \; 20 \cong 2\times 10^4 .$$

Since Γ_{K_3} is approximately 6×10^6 sec^{-1}, the three-body partial width is of the order

$$\Gamma_{D_3} \sim 10^{11} \text{ sec}^{-1} .$$

This is much larger than our estimate for the two-body leptonic rates, by a factor of 500.

Another interesting example along this line is the $F^+_{\ell 3}$ decay.



The $s\bar{s}$ final state is not an unique physical state. Of the three known pseudoscalar mesons with these quantum numbers, only the η and η' have $s\bar{s}$ components. In fact, one can combine these to find

$$s\,\bar{s} = \frac{1}{\sqrt{3}}\,\eta' - \sqrt{\frac{2}{3}}\,\eta .$$

This gives the prediction

$$\frac{F^+ \to \eta\mu^+\nu}{F^+ \to \eta'\mu^+\nu} = \frac{2/3}{1/3} = 2.$$

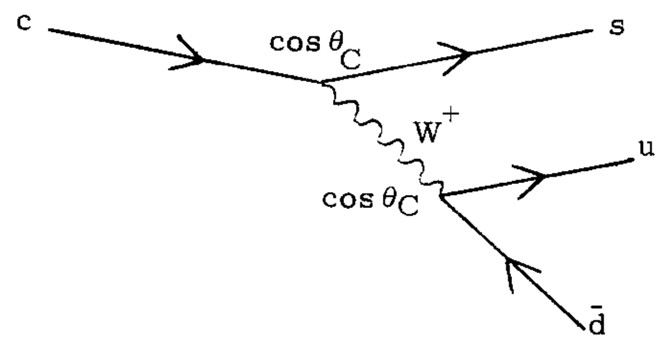
Again, since this reaction is not Cabibbo suppressed, these rates should be comparable to Γ_{D_3} times $(m_F/m_D)^5$, which might not be too large a factor. Another interesting mode of this type is $F^+ \to \phi\mu^+\nu$. This would have a rather distinctive signature, but, as we shall see, these branching ratios are probably quite small (less than 1%).

V. NON-LEPTONIC DECAYS

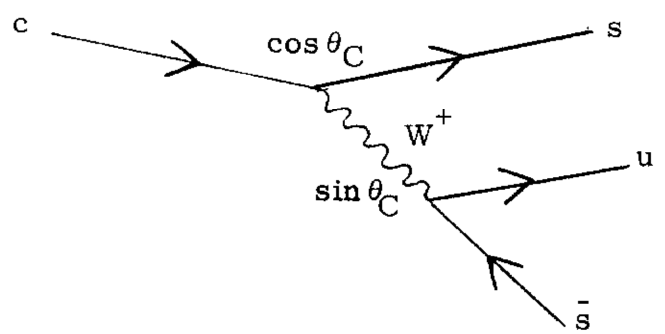
In the preceding chapter, we have discussed the basic weak interaction vertices of quarks and applied them to leptonic and semi-leptonic decays. Now we shall turn to the non-leptonic case, which is more complicated, but also probably more important for charmed particles. The prototype for these processes can be expressed as before, in terms of quark diagrams.

Non-Leptonic Decays

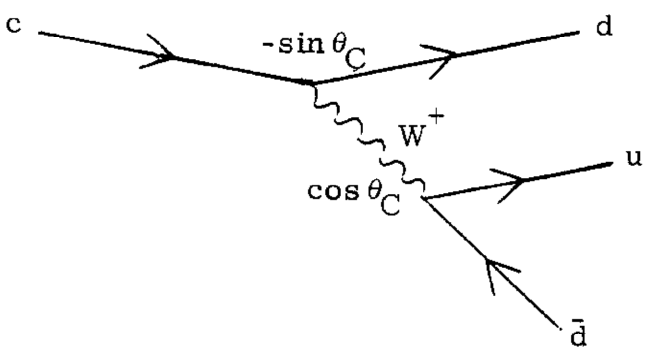
It is easy to draw figures representing nonleptonic decays - just put together any two quark vertices. There are four possibilities for the charmed quark:



amplitude $\propto \cos^2\theta_C$

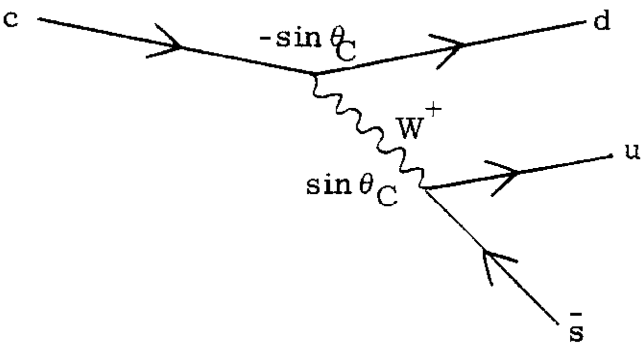


$\propto \cos\theta_C \sin\theta_C$



$$\propto -\cos\theta_C \sin\theta_C$$

and,



$$\propto -\sin^2\theta_C$$

These diagrams represent what one would call the nonleptonic decay of a charmed quark. They do not represent physical processes.

One can apply these ideas to charm-changing decays. For example, GLR have given the following estimate for the non-leptonic decay rate of a charmed particle. As is currently popular, suppose we imagine that a meson looks like a quark-antiquark pair inside a bag. (A bag is a region of space inside of which the quarks are confined.) The non-leptonic decay of the meson may be viewed as the decay of the charmed quark via one of the processes above. The quarks so produced rearrange themselves to form final states of various kinds. (We may also have to allow for creation of other quark-antiquark pairs.) If, in the region inside the bag, the quarks act as if they were nearly free particles in a potential well, then perhaps the total decay rate is of the same order as the primary process $c \rightarrow su\bar{d}$. Let's calculate this rate. It is easily done, assuming the charmed quark is much heavier than the non-charmed quarks. Then, by lepton-hadron universality, the amplitude for $c \rightarrow su\bar{d}$ is simply $\cos^2\theta_C$ times the leptonic amplitude $\mu^- \rightarrow \nu_\mu e^- \overline{\nu_e}$. Consequently, the decay rates are simply related by scaling the phase space, i.e.,

$$\Gamma(c \rightarrow su\bar{d}) = \left(\frac{m_c}{m_\mu}\right)^5 \cos^4\theta_C \, \Gamma(\mu \rightarrow e\nu\bar{\nu})$$

If we take $m_c \cong 1.5$ GeV, then we find

$$\Gamma(c \rightarrow su\bar{d}) \sim 3 \times 10^{11} \sec^{-1} \quad .$$

As expected, this is of the same order as the three-body, semileptonic rates estimated earlier. As we shall argue below, some non-leptonic rates are likely to be larger than this estimate by perhaps a factor of five or more. If so, then non-leptonic decays will dominate the branching ratios of charmed particles, with semileptonic decays contributing less than 10%.

To understand the basis for the expected enhancement of certain non-leptonic decays, we must refresh our memories about the phenomenology of strangeness-changing decays, where the enhancement is called "The $\Delta I = \frac{1}{2}$ Rule". I shall not review all the evidence for the $\Delta I = \frac{1}{2}$ rule, but refer you to the excellent recent review by M. K. Gaillard, "Non-Leptonic Decays" in "Textbook on Elementary Particle Physics (M. Nikolic, ed.), to be published.

As we'll review below, we expect the strangeness-changing decays to also change isospin by 1/2 or 3/2. Typically, however, the $\Delta I = 3/2$ amplitude is only 5% or so of the $\Delta I = 1/2$ amplitude. The question arises whether this is because the $\Delta I = \frac{1}{2}$ amplitude is in some sense larger than expected or whether $\Delta I = 3/2$ is strongly suppressed. It seems that both are true. By comparing non-leptonic with semileptonic rates, it seems that the $\Delta I = \frac{1}{2}$ amplitude is perhaps 5 times larger than comparable semileptonic processes and that the $\Delta I = 3/2$ is suppressed by a factor of 5 or so. (One could quarrel with this interpretation a bit, but I believe it represents the most commonly held point of view.)

Even though we do not fully understand this phenomenon, we must ask whether we expect a similar effect in charm-changing decays. To this end, let me review first why the $\Delta I = \frac{1}{2}$ rule is also called "octet enhancement." If we imagine the W-boson to be very massive, then the currents between which it is exchanged are almost at the same point. Thus the effective weak Hamiltonian is often represented as a current-current product

$$H_W \sim \tfrac{1}{2}(J_\mu^\dagger J^\mu + J^\mu J_\mu^\dagger) \quad ,$$

where I've suppressed the Fermi coupling constant. Forgetting about charm for a moment, the hadronic current has only two terms

$$J = \bar{u}d \cos\theta_C + \bar{u}s \sin\theta_C$$

(The space-time structure is not presently of interest.) Now $\bar{u}d$ is that combination of quarks transforming as a π^-; $\bar{u}s$ the K^-. The two terms in the current thus transform under SU_3 like parts of an octet

$$J = (\underset{\sim}{8})_2^1 \cos\theta_C + (\underset{\sim}{8})_3^1 \sin\theta_C \ .$$

The product of currents, then, must transform like some part of $\underset{\sim}{8} \otimes \underset{\sim}{8}$. As we showed in Chapter II , this can be reduced as follows

$$\underset{\sim}{8} \otimes \underset{\sim}{8} = \underbrace{\underset{\sim}{1} \oplus \underset{\sim}{8}_S \oplus \underset{\sim}{27}}_{\substack{\text{symmetric} \\ \text{states}}} \oplus \underbrace{\underset{\sim}{8}_A \oplus \underset{\sim}{10} \oplus \underset{\sim}{10}}_{\substack{\text{antisymmetric} \\ \text{states}}}$$

Since the Hamiltonian is symmetric, $\{J, J^+\} = H_{eff}$, only the symmetric part is relevant. The singlet can not contribute to strangeness-changing decays. If one examines the isospin subgroups of the $\underset{\sim}{8}$ and $\underset{\sim}{27}$, one finds that the strangeness-changing term in $\underset{\sim}{8}$ transforms like isospin $\frac{1}{2}$, while, for the $\underset{\sim}{27}$, there are terms transforming like both $I = \frac{1}{2}$ and $I = 3/2$. Thus, when this Hamiltonian is sandwiched between states, the octet will connect states differing in isospin by $\Delta I = \frac{1}{2}$ and the $\underset{\sim}{27}$ will connect those differing by either $\Delta I = \frac{1}{2}$ or $\Delta I = 3/2$. Consequently, the SU_3 invariant statement of the $\Delta I = \frac{1}{2}$ rule is the dominance of the $\underset{\sim}{8}$ over the $\underset{\sim}{27}$. [As a technical aside, I might indicate here that recently more significant progress has been made in understanding this enhancement. Following an idea of Ken Wilson, Gaillard and Lee (Phys. Rev. Letters <u>33</u>, 108 (1974)) and Altarelli and Maiani (Phys. Letters <u>52B</u>, 351 (1974)) considered an asymptotically-free theory of strong interactions in which colored quarks interact via colored gluons. They showed that the effect of the gluon cloud is to enhance the octet relative to the 27. However, the magnitude of the effect is too small to account for the observed accuracy of the rule and it seems likely that further suppression of the $\underset{\sim}{27}$ (for example, because of the nature of hadronic wave functions) is required.]

Now let us consider charm-changing decays. The charm-changing weak current is now

$$J = -\bar{c}d \sin\theta_C + \bar{c}s \cos\theta_C \quad .$$

These two terms have the quark content of the D^- and F^- respectively. These form part of an SU_3 triplet, so that the full weak current looks like

$$J = -(\underset{\sim}{3})_2^0 \sin\theta_C + (\underset{\sim}{3})_3^0 \cos\theta_C + (\underset{\sim}{8})_2^1 \cos\theta_C + (\underset{\sim}{8})_3^1 \sin\theta_C \quad .$$

That part of the Hamiltonian mediating charm-changing decays is

$$H_{\Delta C=1} = \{J(\Delta C=1),\ J^+(\Delta C=0)\} \quad .$$

This has the SU_3 transformation properties of $\underset{\sim}{3} \otimes \underset{\sim}{8}$ which has the general decomposition

$$\underset{\sim}{3} \otimes \underset{\sim}{8} = \underset{\sim}{3} \oplus \underset{\sim}{6}^* \oplus \underset{\sim}{15}_M \quad .$$

One can show that, for the particular currents entering H_W, $\underset{\sim}{3}$ is absent from the product. The question now is -- which of the two remaining contributions, the $\underset{\sim}{6}^*$ or the $\underset{\sim}{15}_M$, is enhanced. One way to answer this would be to proceed along the lines of Gaillard and Lee and of Altarelli and Maiani to investigate the question in an asymptotically-free gauge theory of SU_3 invariant strong interactions. It is easy to see what the result would be. In order to explain this, I would have to set up some machinery. Suffice it to say that the enhanced term turns out to be the one antisymmetric under exchange of quark fields (of the form $(\psi_d \psi_c \psi_b \psi_{\bar{a}} \bar{\psi}_d \psi_a \bar{\psi}_b \psi_c)$) and this corresponds to the $\underset{\sim}{6}^*$ rather than to the $\underset{\sim}{15}_M$. We'll now give another argument based on SU_4 invariance which leads to the same result, but I wanted to indicate first that <u>the result depends only on SU_3 invariance and not on SU_4 symmetry.</u>

The following argument has been given by Altarelli, Cabibbo, and Maiani and, independently, by Kingsley, Treiman, Wilcek, and Zee. I will sketch how it goes. For more details and references, see a paper by Quigg and me (FERMILAB-Pub-75/21-THY, February, 1975). Let us consider an SU_4 invariant theory, which, indeed, the strong interactions might be at very short distances. We must classify the current in its properties under SU_4 transformations. Obviously, the currents are elements of a $\underset{\sim}{15}$:

$$J = -(\underset{\sim}{15})_0^2 \sin\theta_C + (\underset{\sim}{15})_0^3 \cos\theta_C + (\underset{\sim}{15})_2^1 \cos\theta_C + (\underset{\sim}{15})_3^1 \sin\theta_C \;.$$

Just as in SU_3, the Hamiltonian contains only the symmetric parts of the products of two $\underset{\sim}{15}$'s , which turn out to be

$$\{J, J^+\} = (\underset{\sim}{15} \otimes \underset{\sim}{15})_{sym.} = \underset{\sim}{1} \oplus \underset{\sim}{15}_S \oplus \underset{\sim}{20}_S \oplus \underset{\sim}{84}_S \;.$$

Again the singlet can not generate charm-changing decays. In addition, for the particular currents that appear here, one can show that no states from $\underset{\sim}{15}_S$ contribute. (See e.g., Einhorn and Quigg, <u>op. cit.</u>) Only the $\underset{\sim}{20}_S$ and $\underset{\sim}{84}_S$ are still in contention. Consider the SU_3 subgroups of the SU_4 representations:

$$\underset{\sim}{20}_S \longrightarrow \underbrace{\underset{\sim}{6} \oplus \underset{\sim}{6}^*}_{|\Delta C| = \pm 1} \oplus \underbrace{\underset{\sim}{8}}_{\Delta C = 0}$$

$$\underset{\sim}{84}_S \longrightarrow \underbrace{\underset{\sim}{6} \oplus \underset{\sim}{6}^*}_{|\Delta C| = \pm 2} \oplus \underbrace{\underset{\sim}{3} \oplus \underset{\sim}{15} \oplus \underset{\sim}{3}^* \oplus \underset{\sim}{15}^*}_{|\Delta C| = \pm 1} \oplus$$

$$\underbrace{\oplus \underset{\sim}{1} \oplus \underset{\sim}{8} \oplus \underset{\sim}{27}}_{|\Delta C| = 0} \;.$$

(For the case of interest, the $\Delta C=\pm 2$ term is absent as is the $\underset{\sim}{3} \oplus \underset{\sim}{3}^*$ piece of the $\Delta C=\pm 1$ term.) Examine the pieces which do not change charm. For strangeness-changing decays, we know the octet to be enhanced relative to the $\underset{\sim}{27}$ subgroup, but, since both the $\underset{\sim}{20}$ and $\underset{\sim}{84}$ contain an octet, this alone is not conclusive. However, if the octet in the $\underset{\sim}{84}$ were enhanced, the $\underset{\sim}{27}$ would also be enhanced in the SU_4 limit. The enhanced octet, therefore, must come from the $\underset{\sim}{20}$. [In the asymptotically-free gauge theories investigated in the references cited, the two octets have different symmetry properties. It was verified that the octet having the symmetry of $\underset{\sim}{20}_S$ is enhanced both relative to the octet having the symmetry of $\underset{\sim}{84}$ and relative to the $\underset{\sim}{27}$. Thus the term "octet enhancement" is somewhat ambiguous by itself, which provides the motivation for the complicated aside.]

Therefore, in the SU_4 limit the analogue of the $\Delta I=\frac{1}{2}$ rule is $\underset{\sim}{20}_S$ dominance. In SU_3 language, this means that the charm-changing decays due to that part of the Hamiltonian transforming like a $\underset{\sim}{6}$ or $\underset{\sim}{6}^*$ will be enhanced. I should emphasize again that this result, which we'll call sextet dominance of charm-changing decays, depends only on SU_3 invariance, as our first argument indicated.

How much of an enhancement do we expect? If SU_4 symmetry were approximately correct at short distances, the enhancement of the sextet would be the same as the enhancement of the octet. However, we don't measure H_W directly but only matrix elements thereof. And SU_4 symmetry is a terrible spectroscopic symmetry (the ρ, K^*, ω, ϕ lie in the same multiplet as the $\psi(3.1)$). Consequently, we don't know what the relative magnitude of sextet versus octet enhancement will be. If charmed mesons are found, we will be able to get a rough idea by comparing the semileptonic to the nonleptonic decay rates.

SOME APPLICATIONS OF SEXTET DOMINANCE

Consider the decay of any of the SU_3 triplet of charmed pseudoscalar mesons $|P_C\rangle$ into some collection $|h\rangle$ of noncharmed hadrons:

$$\langle P_C | H_{weak} | h\rangle \ .$$

Because SU_4 is so badly broken, it is unlikely to be useful to work out the relation of these decays to kaon decays which would obtain in the SU_4 symmetry limit. However, SU_3 is a much better symmetry, so we will analyze these decays assuming the validity of SU_3. As discussed in the preceding sections, under SU_3, H_{weak} transforms as the sum of a $\underset{\sim}{6} \oplus \underset{\sim}{6}^*$ and $\underset{\sim}{15}_M \oplus \underset{\sim}{15}_M^*$. However, we argued that matrix elements of the sextet are likely to be large compared to matrix elements of $\underset{\sim}{15}_M$. So in the following <u>we will neglect the pentadecimet.</u> The quantity

$$\langle P_C | H_{weak} \text{ transforms as } \underset{\sim}{3} \otimes \underset{\sim}{6} = \underset{\sim}{8} \oplus \underset{\sim}{10} \ .$$

Thus the final state hadrons $|h\rangle$ must be in an octet or decimet.

<u>Two Pseudoscalars; Two Vectors</u>

Suppose the final state $|h\rangle$ consists of a pair $|P_1P_2\rangle$ of (uncharmed) pseudoscalar mesons. Each pseudoscalar is a member of an octet, so $|P_1P_2\rangle$ transforms as $\underset{\sim}{8} \times \underset{\sim}{8} = \underbrace{\underset{\sim}{1} \oplus \underset{\sim}{8}_S \oplus \underset{\sim}{27}}_{\text{symmetric}} \oplus \underbrace{\underset{\sim}{10} + \underset{\sim}{10}^* + \underset{\sim}{8}_A}_{\text{antisymmetric}}$.

(The decomposition was worked out in Chapter II.) Assuming sextet dominance, we have seen that $\langle P_C | H_{weak}$ transforms as the sum of an octet or decimet, so <u>a priori</u> we would expect there to be three SU_3 invariant amplitudes for the process (coupling to $\underset{\sim}{10}$, $\underset{\sim}{8}_S$, and $\underset{\sim}{8}_A$.) However, the Pauli principle eliminates two of these: The initial state has spin zero, so the final state must have total angular momentum zero. The final two pseudoscalars are therefore in an S-wave, a symmetric spatial wave function. Because they are bosons, they must also be in a symmetric SU_3 state, and hence $\underset{\sim}{8}_A$ and $\underset{\sim}{10}$ are not allowed. The final state is simply a symmetric octet.

Similarly, one can show that the spin $\otimes$ space wave function for two vector mesons is a state having total angular momentum zero must be symmetric under exchange of the particles. So again the SU_3 state must be symmetric which, by sextet dominance, again implies the final state must be $\underset{\sim}{8}_S$.

Therefore, it is simple to work out (or look up) the branching ratios for various combinations of two pseudoscalar or two vector mesons. These are given in the Table on the next page, taken from the paper by Quigg and me, <u>op. cit.</u> [Actually we assumed a nonet symmetry scheme (with magic mixing angles for the vectors) as is suggested by the nonrelativistic quark model. I won't stop here to explain how this works, but it is necessary to use this in order to relate the η' (X^0) to the pseudoscalar octet or to treat the ω and ϕ properly.] For completeness, we have included decays propertional to $\cos^4\theta_C$, $\cos^2\theta_C\sin^2\theta_C$, and $\sin^4\theta_C$, but, of course decays suppressed by $\tan^2\theta_C \approx 6\%$ are rather unimportant. Notice that, while the Cabibbo-dominant D^0 decays always involve kaons, the dominant D^+ and F^+ decays often lead to channels without kaons, such as $\pi^+\eta$ or $\rho^+\omega$. For example, $(F^+ \to \rho^+\omega)/(F^+ \to K^{*+}\overline{K^{*0}}) = 2$.

There are many other channels involving a pseudoscalar-vector pair, three pseudoscalars, perhaps even a baryon-antibaryon pair, and higher

Relative Rates for F^+ Decay[a)]

$\cos^4\theta_C \times$		$\cos^2\theta_C \sin^2\theta_C \times$	
$\pi^+\eta'$	4/9	$K^+\eta'$	4/9
$K^+\overline{K^0}$	1/3	$K^0\pi^+$	1/3
$\pi^+\eta$	2/9	$K^+\pi^0$	1/6
		$K^+\eta$	1/18
$\rho^+\omega$	2/3	$K^{*+}\phi$	1/3
$K^{*+}\overline{K^{*0}}$	1/3	$K^{*0}\rho^+$	1/3
		$K^{*+}\rho^0$	1/6
		$K^{*+}\omega$	1/6

a) Decays of D^+ can be obtained by multiplying each mode by $\tan^2\theta_C$.

Relative Rates for D^0 Decay

$\cos^4\theta_C \times$		$\cos^2\theta_C \sin^2\theta_C \times$		$\sin^4\theta_C \times$	
$\bar{K}^0\eta'$	4/9	$\eta\eta'$	2/3	$K^0\eta'$	4/9
$K^-\pi^+$	1/3	$\pi^-\pi^+$	1/3	$K^+\pi^-$	1/3
$\bar{K}^0\pi^0$	1/6	K^-K^+	1/3	$K^0\pi^0$	1/6
$\bar{K}^0\eta$	1/18	$\pi^0\eta'$	2/9	$K^0\eta$	1/18
		$\eta\eta$	1/6		
		$\pi^0\pi^0$	1/6		
		$\pi^0\eta$	1/9		
$K^{*-}\rho^+$	1/3	$\phi\phi$	2/3	$K^{*+}\rho^-$	1/3
$\bar{K}^{*0}\phi$	1/3	$K^{*+}K^{*-}$	1/3	$K^{*0}\phi$	1/3
$\bar{K}^{*0}\rho^0$	1/6	$\omega\rho^0$	1/3	$K^{*0}\rho^0$	1/6
$\bar{K}^{*0}\omega$	1/6	$\rho^+\rho^-$	1/3	$K^{*0}\omega$	1/6
		$\rho^0\rho^0$	1/6		
		$\omega\omega$	1/6		

multiplicities, some of which are discussed in Einhorn and Quigg, op. cit. Because of the very many channels open, the branching ratio into any given one is likely to be small. (I would be surprised if any were larger than 5 or 10%.) This may make the experimental search in charmed mesons more difficult than one might like. And, for the reasons indicated above and discussed further in our paper, the abundance or nonabundance of kaons in the final state may not settle the matter.

Notice also that there are no $\cos^4\theta_C$ decays of D^+ which couple to a symmetric octet. The relevant term is $\langle D^+ | \underset{\sim}{(6)}^{22} \cos^2\theta_C \sim \underset{\sim}{(3)}^2 \underset{\sim}{(6)}^{22} \cos^2\theta_C$ which contributes only to the decimet. (Exercise: Show this.) As GLR remarked, a decimet is exotic in the usual quark model sense so it may be that decays into the decimet channel are not enhanced in the same way or perhaps by not as large a factor as are decays into the octet final state.

The decimet contributes in principle to certain other decays, such as pseudoscalar-vector pair or three pseudoscalars, which are likely to be as important as two pseudoscalars or two vectors. If charmed particles are ever discovered, it will be quite interested to see how the systematics of nonleptonic decays work out. We may get a better idea how much of the enhancement-suppression phenomenon is due to H_{weak} itself and how much is due to the particles' wave functions.

Suppose that the decimet were not enhanced at all. Then the enhanced nonleptonic decay rate of $D^\pm$ will only be of order $\tan^2\theta_C$ times the nonleptonic width of the $F^\pm$, D^0, or $\overline{D^0}$. Then the semileptonic branching ratio of $D^\pm$ might well be competative with the nonleptonic channels. The dominant semileptonic decays would be $D^+ \to (\overline{K}\, N\pi)^0 \ell^+ \nu_\ell$, where $(\overline{K}N\pi)^0$ denotes a state consisting of a

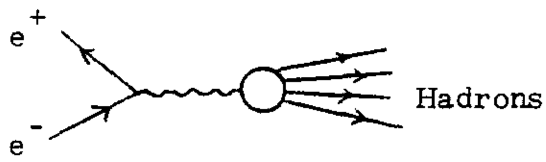
$\overline{K^0}$ or K^- and N pions in a combination having the SU_3 quantum numbers of $\overline{K^0}$.

Although we could spend a great deal more time on the properties of nonleptonic decays, we'll halt our discussion here. In the next lecture, I would like to take up how deep inelastic processes are altered by the presence of charm.

VI. DEEP INELASTIC PROCESSES

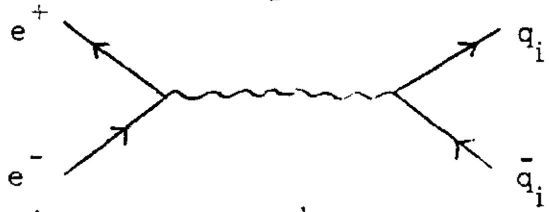
In this lecture, I shall discuss the implications of the charm scheme for deep inelastic processes such as $e^-e^+ \to$ hadrons, and $\ell N \to \ell' X$, where ℓ and ℓ' are any two leptons. I shall not review the theory of these processes but simply recall for you some results from the conventional quark-parton model and then indicate how they are modified by the addition of the charmed

e^-e^+ Annihilation to Hadrons



As $Q^2 \to \infty$, the rate for this process becomes the same, in asymptotically free theories, as if

$$e^-e^+ \to \sum_i q_i \bar{q}_i e_i^2$$



$$R = \left(\frac{e^-e^+ \to \text{hadrons}}{e^-e^+ \to \mu^-\mu^+}\right) = \sum_i e_i^2 .$$

Since $Q = 2/3(c\bar{c} + u\bar{u}) - 1/3(d\bar{d} + s\bar{s})$, we find

$$R = \begin{cases} \left[\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right] 3 = 2 & \underline{\text{below}} \text{ charm threshold} \\ = 2 + \left(\frac{2}{3}\right)^2 3 = \frac{10}{3} & \text{asymptotically} \end{cases}$$

The data are shown in the figure on the next page.

As you can see, one could interpret R as being approximately constant and nearly equal to 2 from $Q^2 \cong 1$ to $Q^2 \cong 9$ GeV^2. There are incredible spikes at 3.1 and 3.7 and a rapid increase to about 6 at the peak at Q = 4.15. It then

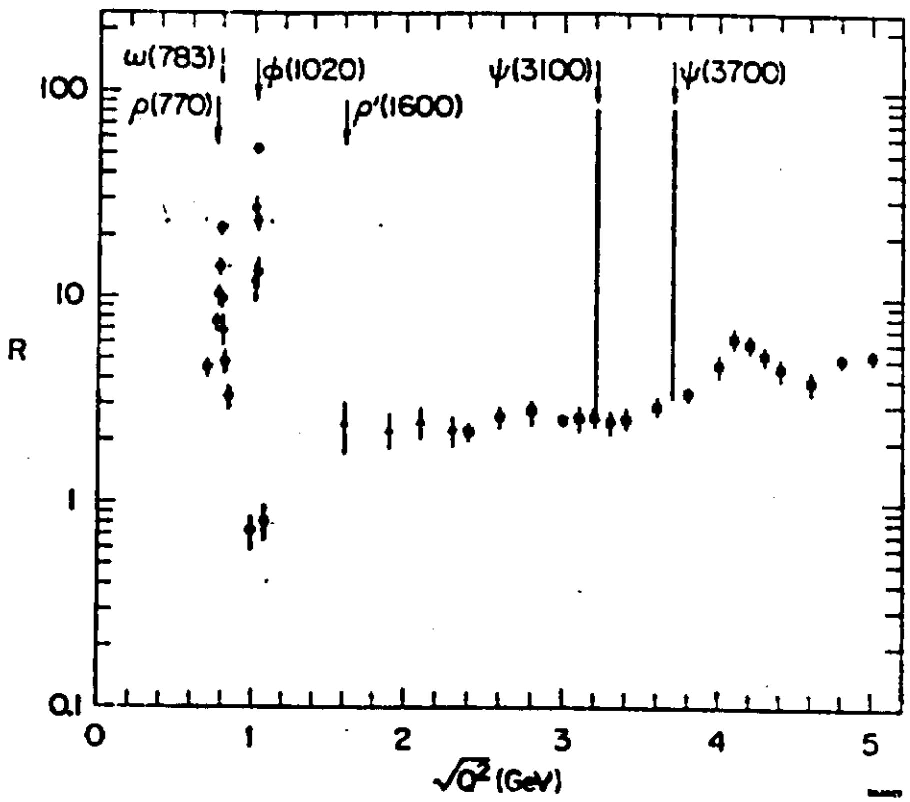


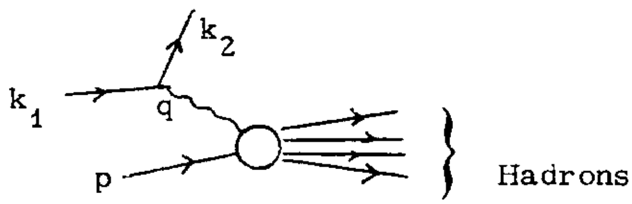
Fig. 1

decreases to around 4 at Q = 4.6 but then increases again to 5 at 4.8 and 5 Gev. These last two points seem troublesome for this picture for asymptotically R must decrease to 10/3. As usual, we would like to see the data at higher energies. One could speculate that the production of pairs of charmed pseudoscalars occurs at 4.1 or so, but other charm channels open at 4.7 or 4.8 (e.g., D^+D^-, $D^0\bar{D}^0$ production at 4.1 and F^+F^- production at 4.7 or charmed vector mesons). Perhaps there are other heavy quarks but the absence of spikes above 3.7 militates against such an interpretation. Let me quit this rank speculation, but I would feel better if R were decreasing, or if the predicted value of R were larger. Alternate schemes having $R \sim 5$ are worth exploring.

Deep Inelastic Lepton Scattering

(See Ben Lee's lectures for derivations of formulas)

Review of Kinematics



$$q \equiv k_1 - k_2 \qquad\qquad \text{Scaling Variables} \quad x \equiv \frac{-q^2}{2M_N\nu} \equiv \frac{1}{\omega}$$

$$M_\nu \equiv q \cdot P \qquad\qquad y \equiv \frac{q \cdot P}{k_1 \cdot P}$$

In Laboratory Frame $P = (M_N, \vec{0})$

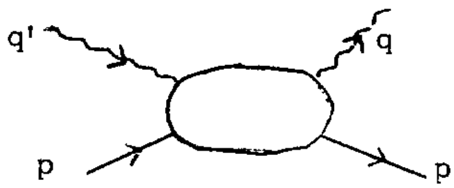
$$q^2 = -4E_1E_2 \sin^2\left(\frac{\theta_L}{2}\right)$$

$$\nu = E_1 - E_2$$

$$x = \frac{2E_1E_2 \sin^2 \frac{\theta_L}{2}}{M_N(E_1 - E_2)} \qquad y = \frac{E_1 - E_2}{E_1}$$

Note: $v = xy = \frac{2E_2}{M_N} \sin^2 \frac{\theta_L}{2}$ is determined entirely by a measurement of the outgoing lepton's energy and direction. This is an especially useful scaling variable when the incident beam energy is unknown (e. g., broad-band neutrino beam).

Because the interaction is mediated by elementary particle exchange (γ or W or Z), the dynamics factorizes into a (known) leptonic piece and an (unknown) hadronic piece. The strong-interaction dynamics is parametrized in terms of structure functions which, by covariance, depend only on q^2, ν, and not, e. g., on $s = (k_1 + p)^2$. In the Bj limit ($\nu \to \infty$, fixed x). These various dimensionless structure functions become functions of x. The dependence on y is determined from kinematics (spin). It has become more-or-less conventional to define the structure functions as follows:



For photon exchange:

$$W_{\mu\nu} = \frac{W_2}{M_N^2}\left(P_\mu + \frac{q_\mu}{2X}\right)\left(P_\nu + \frac{q_\nu}{2X}\right) - W_1\left(q_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)$$

$$\left(\text{Recall } x \equiv -\frac{q^2}{2M_N\nu} = -\frac{q^2}{2P\cdot q}\right).$$

The cross section being related by

$$\frac{d\sigma}{d\Omega dE_2} = \frac{\alpha^2}{4E_1^2 \sin^2\frac{\theta_L}{2}}\left[\cos^2\frac{\theta_L}{2} W_2 + 2\sin^2\frac{\theta_L}{2} W_1\right].$$

For W (or Z) exchange:

$$W_{\mu\nu} = \frac{P_\mu P_\nu}{M_N^2} W_2 - g_{\mu\nu} W_1 - \frac{i\epsilon_{\mu\nu\eta\lambda}}{2M_N^2} P^\eta q^\lambda W_3$$
$$+ \left(\text{terms of order } M_\ell^2\right)$$

All $W_i = W_i(\nu, q^2)$. (All the preceding applies to spinless or unpolarized targets. If polarized, there are other terms.)

Bj Scaling Hypothesis: $(\nu \to \infty$, fixed $x)$

$$W_1 \to F_1(x)$$

$$\frac{\nu W_2}{M_N^2} \to F_2(x)$$

$$\frac{\nu W_3}{M_N^2} \to F_3(x)$$

$$\frac{d\sigma^\nu}{dxdy} \to \frac{G_F^2 M_N E_1}{\pi}\left\{xy^2 F_1 + (1-y)F_2 - xy(1-\tfrac{1}{2}y)F_3\right\}$$

(For $\bar{\nu}$, change sign of F_3 term.)

In <u>principle</u>, these three structure functions could be determined. They simplify for spin $\frac{1}{2}$ constituents: $F_2 = 2xF_1$

$F_3 = \mp 2F_1$ (partons, antipartons)

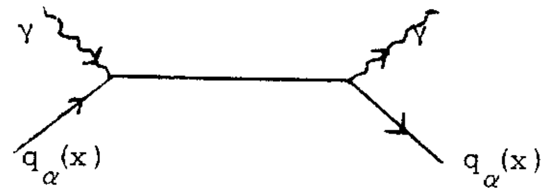
$$\frac{d\sigma^\nu}{dxdy} = \frac{G_F^2 M_N E_1}{\pi}\left\{F_L^\nu(x) + (1-y)^2 F_R^\nu(x)\right\}$$

$$\frac{d\sigma^{\bar{\nu}}}{dxdy} = \frac{G_F^2 M_N E_1}{\pi}\left\{(1-y)^2 F_L^{\bar{\nu}}(x) + F_R^{\bar{\nu}}(x)\right\},$$

where F_L^ν = contribution to F_2^ν from partons (by definition, relative to left-handed neutrino)

F_R^ν = contribution to F_2^ν from antipartons.

This completes our review of kinematics with only spin-one-half constitutents. What if constituents are quarks?



Photon Case

Assume photon is colorless (alternative would be the Han-Nambu scheme, not discussed in these notes). Then, for the proton,

$$\nu W_2 = F_2^{\gamma p} = \left(\frac{2}{3}\right)^2 x \left[U(x) + C(x)\right] + \left(\frac{1}{3}\right)^2 x \; [D(x) + S(x)],$$

where $Q_\alpha(x) \equiv q_\alpha(x) + \bar{q}_\alpha(x)$ for each of the four quarks. $q_\alpha(X)$ = probability to find quark ψ^α with momentum fraction x. Recall

$$q_\alpha(x) \to \frac{C_\alpha}{x} \text{ as x gets wee } \left[0\left(\frac{1}{\sqrt{\nu}}\right)\right].$$

If SU_3 symmetry holds, $C_1 = C_2 = C_3$ if $u = d = s = \bar{u} = \bar{d} = \bar{s}$. If SU_4 symmetry (bad), charmed quarks are also equal.

So long as the probability of finding a charmed quark in a proton is small (compared to noncharmed quarks), charm will have no effect on deep inelastic electroproduction. (If it has any effect at all, it would be expected to occur for x small.) This is quite an important point. Even though charm causes dramatic changes in e^-e^+ annihilation (and also the production of $\mu^-\mu^+$ pairs in hadronic collisions), this behavior is reconcilable with scaling at low energies in electroproduction. It simply means that there are few charmed quarks to be found in a nucleon. This means, for example, that although the $\phi_c(\psi)$ is easily produced in photoproduction (and probably in electroproduction at small q^2), it will probably disappear at large q^2. This should be quite analogous to rho photoproduction. We do not expect deep inelastic electro-production to be a good source of ϕ_c's or particles of non-zero charm. As we shall soon see, the situation may be very different in neutrino scattering.

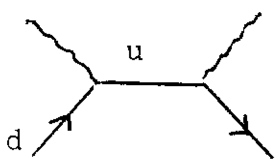
$$F_2^{\gamma n} = \left(\frac{2}{3}\right)^2 x\,[D(x) + C(x)] + \left(\frac{1}{3}\right)^2 x\,\left[U(x) + S(x)\right]$$

$$F_2^{\gamma} \equiv F_2^{\gamma p} + F_2^{\gamma n} = \frac{5}{9}\,x\left[U(x) + D(x)\right] + \frac{8}{9}\,x\,C(x) + \frac{2}{9}\,x\,S(x).$$

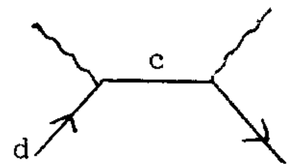
Neutrino Scattering (Charged Current):

What Can Happen?

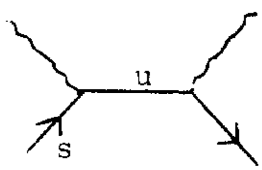
Without Charm | With Charm



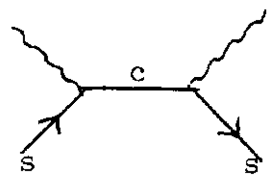
$d(x)\cos^2\theta_C$



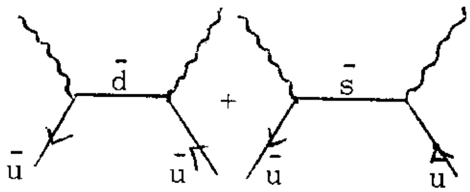
$d(x)\sin^2\theta_C$



$s(x)\sin^2\theta_C$

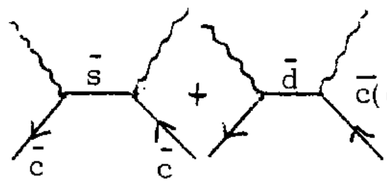


$s(x)\cos^2\theta_C$



$\bar{u}(\cos^2\theta_C + \sin^2\theta_C)$

$= \bar{u}(x)$



$\bar{c}(\cos^2\theta_C + \sin^2\theta_C)$

$= \bar{c}(x)$

Recall

$$\frac{d\sigma}{dxdy} \propto \begin{cases} 1 \text{ for } \nu\text{-}q & (\text{or } \bar{\nu}\text{-}\bar{q}) \\ (1-y)^2 \text{ for } \nu\text{-}\bar{q} & (\text{or } \bar{\nu}\text{-}q). \end{cases}$$

Thus

$$\frac{d\sigma^{\nu p}}{dxdy} = \frac{G_F^2 M_N E_1}{\pi}\left\{F_L^{\nu p} + (1-y)^2 F_R^{\nu p}\right\},$$

$$\frac{d\sigma^{\bar{\nu} p}}{dxdy} = \frac{G_F^2 M_N E_1}{\pi}\left\{F_L^{\bar{\nu} p}(1-y)^2 + F_R^{\bar{\nu} p}\right\}$$

where,

Without Charm	With Charm
$F_L^{\nu p} = d(x)\cos^2\theta_C + s(x)\sin^2\theta_C$	$d(x) + s(x)$
$F_R^{\nu p} = \bar{u}(x)$	$\bar{u}(x) + \bar{c}(x)$

For antineutrinos, simply interchange role of quarks and antiquarks, so

Without Charm	With Charm
$F_L^{\bar{\nu} p} = u(x)$	$u(x) + c(x)$
$F_R^{\nu p} = \bar{d}(x)\cos^2\theta_C + \bar{s}(x)\,\sin^2\theta_C$	$\bar{d}(x) + \bar{s}(x)$

For a neutron, simply interchange $\frac{u \leftrightarrow d}{\bar{u} \leftrightarrow \bar{d}}$. For matter (equal mixtures of protons and neutrons) define

$$F_i^{\nu} = F_i^{\nu p} + F_i^{\nu n} \qquad i = L, R$$

Similarly for $F_i^{\bar{\nu}}$. Then

Without Charm	With Charm
$F_L^{\nu} = (u + d)\cos^2\theta_C + 2s\,\sin^2\theta_C$	$u + d + 2s$
$F_R^{\nu} = \bar{u} + \bar{d}$	$\bar{u} + \bar{d} + 2\bar{c}$
$F_L^{\bar{\nu}} = u + d$	$u + d + 2c$
$F_R^{\bar{\nu}} = (\bar{u} + \bar{d})\cos^2\theta_C + 2\bar{s}\,\sin^2\theta_C$	$\bar{u} + \bar{d} + 2\bar{s}$

Valence quarks only: $(\bar{u} = \bar{d} = \bar{s} = s = c = \bar{c} = 0)$

$F_L^{\nu} = F_L^{\bar{\nu}}$ "Charge Symmetry Invariance"

$F_R^{\nu} = F_R^{\bar{\nu}} = 0$

In the absence of charm, because of the smallness of the Cabibbo angle, charge symmetry may work pretty well. If

$$2s\,\tan^2\theta_c << u + d$$
$$2\bar{s}\,\tan^2\theta_c << \bar{u} + \bar{d},$$

then $F_L^{\nu} \approx F_L^{\bar{\nu}}$ and $F_R^{\nu} \approx F_R^{\bar{\nu}}$ (even though non-zero). Because $\tan^2\theta_c \approx 6\%$, we expect that, even if $2\bar{s} \approx \bar{u} + \bar{d}$ and $2s \approx u + d$, as would be the case in the SU_3 symmetry limit, charge symmetry will hold to about 5% in the absence of charm. We discuss this further later in this lecture.

Time Out for Data

Suppose contributions due to charm could be neglected, e.g., because we are below threshold. Because the Cabibbo angle is small, we can probably neglect contributions due to strange quarks as well. This enables us to determine u+d and $\bar{u}+\bar{d}$ from data on neutrinos and antineutrinos. Based on the data from Gargamelle, Perkins has deduced these (Fig. 1). (See his lectures at Hawaii Summer School.) Notice that the data are consistent with the valence quark picture for $x \gtrsim 0.3$ or so, i.e., $\bar{u} + \bar{d} = 0$. (Nothing can be said about s and $\bar{s}$ because of the small Cabibbo angle.) For $x < 0.3$, it would appear that $\bar{u} + \bar{d} \neq 0$, but it is questionable whether these data are relevant to scaling for $x \lesssim 1/4$, because neutrinos as low as 2 GeV are included (so $Q^2 < 1\ \mathrm{GeV}^2$). Given these values of $u + d$ and $\bar{u} + \bar{d}$, we can go further. From SLAC, we know $\nu W_2^{\gamma p}$ and $\nu W_2^{\gamma n}$. One can combine all these data in another way to deduce $u + \bar{u}$, $d + \bar{d}$, and $s + \bar{s}$ (there is no Cabibbo suppression in electroproduction). The resulting points are shown in Fig. 2 [from Savit and Einhorn, Phys. Rev. Letters 33, 392 (1974)]. Notice that, although the errors are large, it would appear that $S = s + \bar{s}$ is non-zero for $x \lesssim 0.6$. (We do not show S for $x \lesssim 0.2$ where it becomes large and negative. I believe this is a reflection

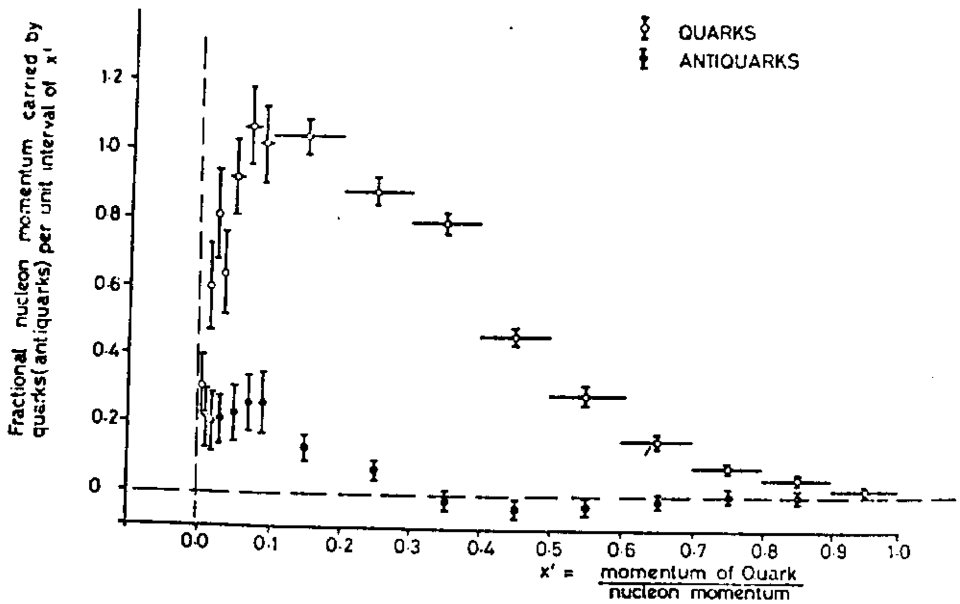

QUARKS
ANTIQUARKS
Fractional nucleon momentum carried by quarks(antiquarks) per unit interval of x'
1·2
1·0
0·8
0·6
0·4
0·2
0
0·0
0·1
0·2
0·3
0·4
0·5
0·6
0·7
0·8
0·9
1·0
x' = momentum of Quark / nucleon momentum

Fig. 1

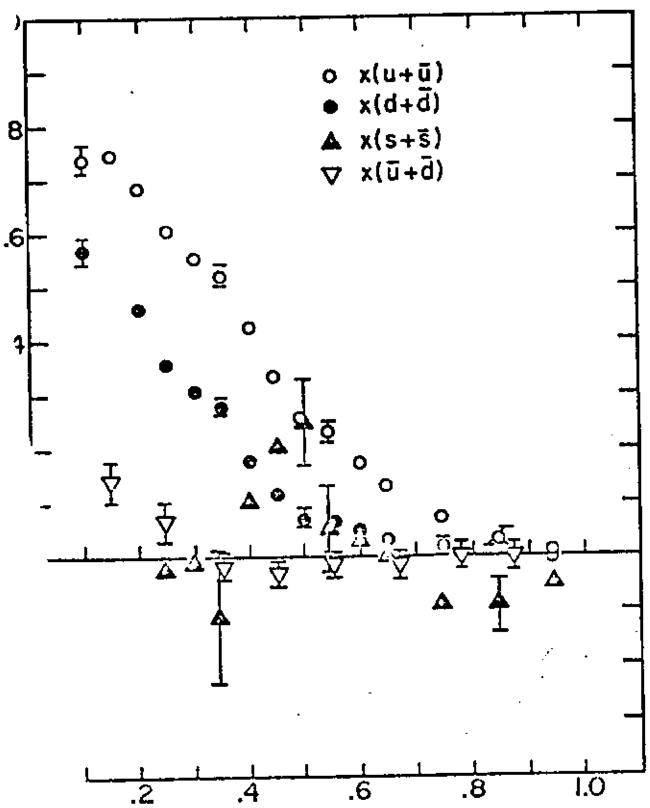

x(u+ū)
x(d+d̄)
x(s+s̄)
x(ū+d̄)
.8
.6
.2
.4
.6
.8
1.0

Fig. 2

of the non-scaling character of the small x neutrino data rather than a fundamental breakdown of the underlying picture.)

Returning to theory, the so-called "charge symmetry" relation receives a lot of attention, so let me discuss it a bit further. This relation comes about in the absence of charm by setting $\theta_C = 0$. Then $F_i^{\nu p} = F_i^{\bar{\nu} n}$ (i = R, L). We have seen that, in the absence of charm, we expect it to hold to an accuracy of 5% or so, even better so long as the valence quarks dominate. Now consider the inclusion of charmed quarks. Can we still have $F_L^{\nu p} = F_L^{\bar{\nu} n}$? Looking back at our formulas, this would mean that $d(x) + s(x) \approx d(x) + c(x)$, which would be true only if either

(1) $s(x) \approx c(x)$ (Unlikely, I believe except possibly at $x=0$) or,

(2) $s(x) << d(x)$ and $c(x) << d(x)$.

It is easy to believe that $c << d$ even at small x since SU_4 is a badly broken symmetry. However, SU_3 is much better and it is unlikely that $s(x) << d(x)$ at least at small x (say $x \lesssim 0.1$). Therefore, with charm, we might expect a breakdown of charge symmetry invariance at small x. In experiments on nuclei (approximately equal mixtures of protons and neutrons), we would have $F_L^{\nu} \approx F_L^{\bar{\nu}}$ if either

(1) $s(x) \approx c(x)$ or

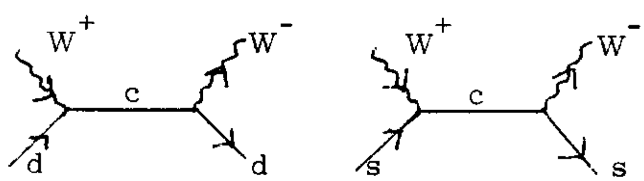
(2) $s(x) << \frac{1}{2}(u + d)$ and $c(x) << \frac{1}{2}(u + d)$.

Turning to the antiquarks, we would have $F_R^{\nu} \approx F_R^{\bar{\nu}}$ if $u + d + 2c \approx \bar{u} + \bar{d} + 2\bar{s}$ which would obtain if either (1) $\bar{c} \approx \bar{s}$ or (2) $2\bar{c}$, $2\bar{s} << \bar{u} + \bar{d}$. It is even more difficult to believe that $2\bar{s} << \bar{u} + \bar{d}$ than to believe $2s << u + d$. Thus we are even more confident that, with charm, for small x (say $x \lesssim 0.2$), charge symmetry invariance will surely fail for the right-handed structure functions, F_R.

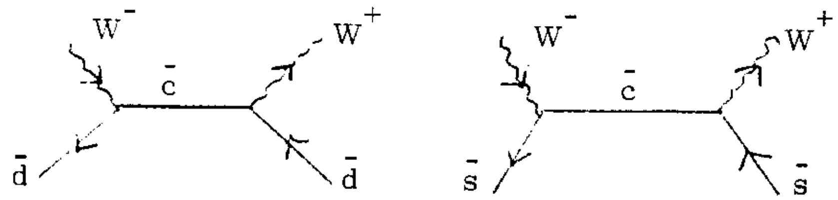
Remarks on recent data:

There is some indication from Experiment #1A (Cline-Mann-Rubbia) that, for $x \lesssim 0.1$, there is a breakdown of charge symmetry invariance, especially for antineutrinos. [See Phys. Rev. Letters 34, 597 (1975) and references therein.] The breakdown occurs only for E_ν or $E_{\bar{\nu}} \gtrsim 30$ GeV, as if there is some threshold effect. It is tempting to interpret this as charm. Such an interpretation will depend on further data with better statistics so that careful determinations of the quark distributions would be possible. With reasonable quark distributions, I have not been able to understand their data, assuming scaling. It is likely that threshold effects are important, however.

Back to theory: Without getting into a long controversial discussion of what these quark-parton diagrams have to do with final states, let me speculate on how charm will show up in final states in neutrino scattering. Let me assume that I can neglect charmed quarks in a nucleon ($c = \bar{c} = 0$) compared to non-charmed quarks. Let me further assume that, when a charmed quark is produced, it somehow always evolves into a charmed hadron. Then for charm production in neutrino scattering, the important diagrams (for νp scattering) are



$d(x)\sin^2\theta_C + s(x)\cos^2\theta_C$, contributing to constant term in y; in antineutrino ($\bar{\nu}p$) scattering, they are



$\bar{d}(x)\sin^2\theta_C + \bar{s}(x)\cos^2\theta_C$, contributing to constant term in y. To summarize, we are guessing that for p + n or Fe or linoleum)

$$\frac{d\sigma^{\nu}}{dxdy} \propto \begin{cases} [(u+d)\cos\theta_C^2 + 2s\sin\theta_C^2] + (1-y)^2[\bar{u}+\bar{d}] & \text{leads to } C = 0 \text{ final state} \\ [(u+d)\sin\theta_C^2 + 2s\cos\theta_C^2] & \text{leads to } C = +1 \text{ final state} \end{cases}$$

$$\frac{d\sigma^{\bar{\nu}}}{dxdy} \propto \begin{cases} [(\bar{u}+\bar{d})\cos\theta_C^2 + 2\bar{s}\sin\theta_C^2] + (1-y)^2[u+d] & \text{leads to } C = 0 \text{ final state} \\ [(\bar{u}+\bar{d})\sin\theta_C^2 + 2\bar{s}\cos\theta_C^2] & \text{leads to } C = -1 \text{ final state} \end{cases}$$

Neutrino scattering will lead to a hadron with C = 1, perhaps a charmed baryon (e.g., C_1^+ or C_1^{++}) or a charmed meson (F^+, D^+, or D^0). Antineutrino scattering will lead to C = -1 hadron states. This could be an antibaryon but, if threshold effects play as significant a role as we suspect, this would be much suppressed so we would only find charmed mesons (e.g., F^-, D^-, D^0).

We have argued previously (Lecture V) that the semileptonic branching ratios would be less than 1% for most charmed hadrons, with the possible exception of $D^{\pm}$. In any case, the underlying process is $c \to s\mu^+\nu_\mu$ (in C = +1 hadrons) or $\bar{c} \to \bar{s}\mu^-\bar{\nu}_\mu$ (for C = -1 hadrons). There would then be two muons in the final state, necessarily of opposite sign. Is this the source of 14 dimuons of opposite sign recently reported? [See Benvenuti et al., Phys. Rev. Letters 34, 419 (1975).] More events of this type have been seen (D. Cline, private communication).

Even in the absence of positive identification of a charmed hadron, how would crossing charm threshold lead to a change in the deep inelastic structure functions? In neutrino scattering, we see that it would cause an increase in the constant term in (1 - y) in $d\sigma^{\nu}/dxdy$. The percentage increase is likely to be small, since this constant term is dominated by valence quarks. One may need to go to quite small values of x to see it.

In antineutrino scattering, the charmed final states again contribute to a constant term in (1 - y). The occurrence of a constant term is expected anyway, due to non-valence quark contributing to C = 0 final states. So the presence of a constant piece is no surprise. However, above charm threshold, the constant would increase, probably by a significant amount. Obviously, this would be most easily seen near y = 1.

It is sometimes remarked that it should be easier to produce charm in neutrino scattering than in antineutrino scattering because the neutrino need only find a valence quark while the antineutrino must hit a non-valence quark. Let me caution you against such an interpretation. As you can see, the contribution of valence quarks in $d\sigma^{\nu}$ which lead to charmed final states is suppressed by $\sin^2\theta_C$, whereas strange quarks produced charmed quarks with $\cos^2\theta_C$ for both neutrinos and antineutrinos. To say this again in the context of an oversimplified model, consider the valence quark + SU_3-invariant sea, but suppose the probability of finding charmed quarks in nucleons is negligible:

$$u = u_v(x) + q_0(x) \qquad\qquad \bar{u} = \bar{d} = s = \bar{s} = q_0(x)$$

$$d = d_v(x) + q_0(x) \qquad\qquad c = \bar{c} = 0$$

Then

$$\frac{d\sigma^{\nu}}{dxdy} \propto \begin{cases} [(u_v + d_v)\cos^2\theta_C + 2q_0] + 2q_0(1-y)^2 & \text{leads to } C = 0 \\ (u_v + d_v)\sin^2\theta_C + 2q_0 & \text{leads to } C = 1 \end{cases}$$

$$\frac{d\sigma^{\bar{\nu}}}{dxdy} \propto \begin{cases} 2q_0 + (u_v + d_v + 2q_0)(1-y)^2 + 2q_0 & \text{leads to } C = 0 \\ 2q_0 & \text{leads to } C = -1 \end{cases}$$

Whether it is significantly easier to produce charm in ν scattering depends on the relative magnitude of $[(u_v + d_v)/2]\sin^2\theta_c$ and q_0. For small x, I suspect q_0 is quite comparable to the valence quark contribution, so I would guess that neutrinos offer no particular advantage over antineutrinos for producing charmed final states.

This concludes our introduction to SU_4 and charm, as I have over-run the time allotted. I hope these lectures help some of you to better appreciate the burdgening literature on this subject. I encourage you, theorists or experimenters, to come see me or other members of Fermilab's Theoretical Group if you have any questions at all. You might have precisely the question none of us has thought to ask!