

**COURSE 3**  
**GAUGE THEORIES**

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## 1. Yang-Mills fields

### 1.1. Introductory remarks

Professor Faddeev has discussed the quantization problem of a system which is described by a singular Lagrangian. For the following, we shall assume that the student is familiar with the path integral formalism, and the quantization of the Yang-Mills theory. The following remarks are intended to agree on notations.

The Yang-Mills Lagrangian, without matter fields, may be written as

$$\mathcal{L} = -\frac{1}{4} [\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f_{abc} A_\mu^b A_\nu^c]^2. \quad (1.1)$$

For simplicity we shall assume the underlying gauge symmetry is a simple compact Lie group  $G$ , with structure constants  $f_{abc}$ .

The Lagrangian (1.1) is invariant under the gauge transformations,

$$\begin{aligned} L_a A_\mu^a(x) &\rightarrow [L_a A_\mu^a(x)]^\epsilon \\ &= U(\epsilon) [L_a A_\mu^a(x) - \frac{i}{g} U^{-1}(\epsilon) \partial_\mu U(\epsilon)] U^{-1}(\epsilon), \end{aligned} \quad (1.2)$$

where the  $\epsilon$  are space-time dependent parameters of the group  $G$ ,  $U^{-1}(\epsilon) = U^\dagger(\epsilon)$  and the  $L$  are the generators.

These gauge transformations form a group, i.e., if  $g'g = g''$ , then

$$A_\mu \xrightarrow{g} A_\mu^g \xrightarrow{g'} (A_\mu^g)^{g'} = A_\mu^{g'g} = A_\mu^{g''}.$$

(Problem: prove this statement.)

The infinitesimal version of the gauge transformation is

$$L_a \delta A_\mu^a = -\frac{1}{g} L_c \partial_\mu \epsilon^c - f_{abc} A_\mu^a \epsilon^b L^c,$$

or

$$\delta A_\mu^a = -\frac{1}{g} \partial_\mu \epsilon^a + f_{abc} \epsilon^b A_\mu^c. \quad (1.3)$$

It is precisely this freedom of redefining fields without altering the Lagrangian that lies in the heart of the subtlety in quantizing a gauge theory. In the language of the operator field theory, to quantize a dynamical system one has to find a set of initial value variables,  $p$ 's and  $q$ 's, which are complete, in the

sense that their values at time zero determine the values of these dynamical variables at all times. It is only in this case that the imposition of canonical commutation relations at time zero will determine commutators at all times and define a quantum theory for a gauge theory. This can never be done because we can always make a gauge transformation which vanishes at time zero. That is, it is impossible to find a complete set of initial-value variables in a gauge theory unless we remove this freedom of gauge transformations.

To quantize a gauge theory, it is necessary to choose a gauge, that is, impose conditions which eliminate the freedom of making gauge transformations, and see if a complete set of initial-value variables exist.

There is a special gauge, called the axial gauge, in which the quantization is particularly simple. It is defined by the gauge condition that

$$\eta^\mu A_\mu^a(x) = 0, \quad (1.4)$$

where  $\eta$  is an arbitrary four-vector. In this gauge, the vacuum-to-vacuum amplitude can be written as

$$e^{iW} = N \int [dA_\mu^a] \prod_{a,x} \delta(A^a(x) \cdot \eta) \quad (1.5)$$

$$\times \exp[i \int d^4x \{ -\frac{1}{4} (\partial_\mu A_\nu^a(x) - \partial_\nu A_\mu^a(x) + f_{abc} A_\mu^b(x) A_\nu^c(x))^2 \}],$$

where  $N$  is a normalizing factor.

There is in principle no reason why eq. (1.5) cannot be used to generate Green functions, by the usual device of adding a source term in the action. That is, we define the generating functional of the connected Green functions  $W_A [J_\mu^a]$  by

$$e^{iW_A [J_\mu^a]} = N \int [dA_\mu^a] \prod_{a,x} \delta(\eta \cdot A^a(x)) \quad (1.6)$$

$$\times \exp[i \int d^4x \{ \mathcal{L}(x) + J_\mu^a(x) A^{\mu a}(x) \}].$$

However, the Feynman rules would not be manifestly Lorentz covariant in this gauge and it is desirable to develop quantum theory of the Yang-Mills fields in a wider class of gauges.

As Professor Faddeev explained, eq. (1.6) is an injunction that the path integral is to be performed not over all variations of  $A_\mu^a(x)$ , but over distinct orbits of  $A_\mu^a(x)$  under the action of the gauge group. To implement this idea, a

“hypersurface” was chosen by the gauge condition  $\eta \cdot A^a = 0$ , so that the hypersurface in the manifold of all field intersects each orbit only once. The problem we pose ourselves is how to evaluate eq. (1.6) if we are to choose a hypersurface other than the one for the axial gauge.

1.2. Problem: Coulomb gauge

The gauge defined by  $\nabla_i A_i^a = 0$  is called the Coulomb gauge. In this gauge the two space-like transverse components of  $A_\mu^a$  are the  $q$ 's, and the two space-like transverse components of  $F_{0i}^a$  are the  $p$ 's,

$$F_{0i}^a = F_{0i}^{aT} + \nabla_i f^a,$$

$$\nabla_i F_{0i}^T = 0.$$

Express the Lagrangian (1.1) in terms of the Coulomb gauge variables  $A_i^a$ ;  $\nabla_i A_i^a = 0$ ;  $F_{0i}^{aT}$ ;  $f^a$ ;  $F_{ij}^a$ ;  $A_0^a$ . Referring to Professor Faddeev's lecture, compute  $\Delta(p, q)$  for this gauge.

$\Delta(p, q)$



2. Perturbation expansion for quantized gauge theories

2.1. General linear gauges

The foregoing example, the axial gauge condition, is but one of the ways to eliminate the possibility of gauge transformations during the period the temporal development of a quantized system of gauge fields is studied. Clearly this is not the only way, and in fact, we could define a gauge by the equation

$$F^a[A_\mu^b, \varphi] = 0 \quad \text{for all } a, \tag{2.1}$$

provided that, given  $A_\mu^b$ , and other fields which we shall collectively call  $\varphi$  there is one and only one gauge transformation which makes eq. (2.1) true. For convenience, we shall deal only with the cases in which  $F^a$  is linear in the boson fields  $A_\mu^a$  and  $\varphi$ . In this lecture, however, we shall be concerned primarily with the instance in which  $F^a$  depends on  $A_\mu^a$  alone.

Before proceeding further, let us pause here briefly to review a few facts about group representations. Let  $g, g' \in G$ . Then  $gg' \in G$  and

$$U(g)U(g') = U(gg'). \tag{2.2}$$

The invariant Hurwitz measure over the group  $G$  is an integration measure of the group manifold which is invariant in the sense that

$$dg' = d(g'g). \quad (2.3)$$

If we parametrize  $U(g)$  in the neighborhood of the identity as

$$U(g) = 1 + i\epsilon^a L_a + O(\epsilon^2), \quad (2.4)$$

then we may choose

$$dg = \prod_a d\epsilon^a \quad g \simeq 1. \quad (2.5)$$

Consider now the integral

$$\Delta_F^{-1}[A_\mu^b] = \int \prod_x dg(x) \prod_{a,x} \delta(F^a[(A_\mu^b(x))^g]), \quad (2.6)$$

where  $(A_\mu^b(x))^g$  denotes the  $g$ -transform of  $A_\mu^b(x)$ , as defined by eq. (1.5). The quantity  $\Delta_F[A_\mu^b]$  is gauge invariant, in the sense that

$$\begin{aligned} \Delta_F^{-1}[(A_\mu^b)^g] &= \int \prod_x dg'(x) \prod_{a,x} \delta(F^a[(A_\mu^b(x))^{g'g}]) \\ &= \int \prod_x d(g'g)(x) \prod_{a,x} \delta(F^a[(A_\mu^b(x))^{g'g}]) \\ &= \int \prod_x dg''(x) \prod_{a,x} \delta(F^a[(A_\mu^b(x))^{g''}]) \\ &= \Delta_F^{-1}[A_\mu^b], \end{aligned} \quad (2.7)$$

where we made use of eq. (2.3).

According to eq. (1.5), we can write the vacuum-to-vacuum amplitude as

$$e^{iW} = N \int [dA_\mu^a] \prod_{a,x} \delta(\eta \cdot A^a(x)) e^{iS[A]}, \quad (2.8)$$

where  $S = \int d^4x \mathcal{L}(x)$  is the action. Since

$$\Delta_F[A] \int \prod_x dg(x) \prod_{a,x} \delta(F^a[A_\mu^g(x)]) = 1, \quad (2.9)$$

we may rewrite eq. (2.8) as

$$e^{iW} = N \int [dA] \Delta_F[A] \int \prod_x dg(x) \prod_{a,x} \delta(F^a[A^g(x)]) \\ \times \prod_{b,y} \delta(\eta \cdot A^b(y)) \exp\{iS[A]\}. \quad (2.10)$$

In the integrand we can make a gauge transformation  $A_\mu^b(x) \rightarrow (A_\mu^b(x))^{g^{-1}}$ . Under the gauge transformation of eq. (1.5), the metric  $[dA]$ , the action  $S[A]$  and  $\Delta_F[A]$  remain invariant, so eq. (2.10) may be written as

$$e^{iW} = N \int [dA] \Delta_F[A] \prod_{a,x} \delta(F^a[A(x)]) e^{iS[A]} \\ \times \int \prod_y dg(y) \prod_{b,y} \delta((\eta \cdot A^b(y))^{g^{-1}}). \quad (2.11)$$

Let us assume that

$$(\eta \cdot A^b(y))^{g_0} = 0.$$

Then we have

$$\int \prod_y dg(y) \prod_{b,y} \delta((\eta \cdot A^b(y))^{g^{-1}}) = \int \prod_y dg'(y) \prod_{b,y} \delta(((\eta \cdot A^b)^{g_0})^{g'}) \\ = \int \prod_{a,y} d\epsilon^a(y) \prod_{b,y} \delta\left(\frac{1}{g} \eta \cdot \partial \epsilon^b(y)\right),$$

which is a constant independent of  $A$ . Therefore, this constant may be absorbed in  $N$ , and

$$e^{iW} = N \int [dA] \Delta_F[A] \prod \delta(F^a[A]) e^{iS[A]}. \quad (2.12)$$

This is the vacuum-to-vacuum amplitude evaluated in the gauge specified by eq. (2.1).

Let us evaluate  $\Delta_F[A]$ . Since in eq. (2.12) this is multiplied by  $\prod \delta(F_a[A])$ , we need only to know  $\Delta_F[A]$  for  $A$  which satisfies eq. (2.1). Let us make a

gauge transformation on  $A$  so that  $F_a[A] = 0$ . For  $g$  in the neighborhood of the identity, then

$$\begin{aligned} F^a[A^g] &= F^a[A] + \frac{\partial F^a}{\partial A_\mu^b} \left( -\frac{1}{g} \partial_\mu \delta_{b,c} + f_{bcd} A_\mu^d \right) \epsilon^c \\ &= -\frac{\partial F^a}{\partial A_\mu^b} (D_\mu \epsilon)^b \frac{1}{g}, \end{aligned} \quad (2.13)$$

where  $D_\mu$  is the covariant derivative

$$D_\mu^{ab} = \delta_{a,b} \partial_\mu - g f_{abc} A_\mu^c. \quad (2.14)$$

Therefore, from eq. (2.6), we see that

$$\begin{aligned} \Delta_F^{-1}[A] &= \int \prod_{a,x} d\epsilon^a(x) \prod_{b,y} \delta \left[ \frac{\partial F^a}{\partial A_\mu^b} D_\mu^{bc} \epsilon_c(x) \right] \\ &\sim [\det M_F]^{-1}, \end{aligned}$$

where

$$\langle a, x | M_F | b, y \rangle = \frac{\partial F^a}{\partial A_\mu^c} D_\mu^{cb} \delta^4(x - y)$$

or

$$\sim \frac{\delta F^a[A(x)]}{\delta \epsilon^b(y)}. \quad (2.16)$$

That is,

$$\Delta_F[A] \sim \det M_F \sim \det \left( \frac{\delta F^a(x)}{\delta \epsilon^b(y)} \right). \quad (2.17)$$

Here, we can afford to be sloppy about the normalization factors, as long as they do not depend on the field variables  $A_\mu$ .

The factor  $\Delta_F[A]$  can be evaluated from eq. (2.17) for various choices of  $F$ . The example we will consider is the so-called Lorentz gauge,

$$F^a = \partial^\mu A_\mu^a + c^a(x),$$

where  $c^a(x)$  is an arbitrary function of space-time. Under the infinitesimal gauge transformation (1.3) of  $A_\mu^a$ ,  $F_a$  changes by

$$\delta F^a = -\frac{\partial^\mu}{g}(\delta_{ab} \partial_\mu - gf_{abc} A_\mu^c) \epsilon^b,$$

so that

$$\langle a, x | M_L | b, y \rangle = -\partial^\mu D_\mu^{ab} \delta^4(x - y).$$

The appearance of the delta functional  $\Pi \delta(F_a[A])$  makes eq. (2.12) not very amenable to practical calculations. We could have chosen as gauge condition:

$$F^a[A] - c^a(x) = 0, \tag{2.18}$$

with an arbitrary space-time function  $c_a$ , instead of eq. (2.1). The determinant  $\Delta_F[A]$  is still the same as before, that is, is given by eq. (2.17), and clearly the left-hand side of eq. (2.12) is independent of  $c_a$ . Thus, we may integrate the right-hand side of the equation

$$e^{iW} = N \int [dA] \Delta_F[A] \prod \delta(F^a[A] - c^a) e^{iS[A]} \tag{2.19}$$

over  $c_a(x)$  with a suitable weight, specifically with

$$\exp\left(\frac{-i}{2\alpha} \int d^4x c_a^2(x)\right),$$

where  $\alpha$  is a real parameter, and obtain

$$e^{iW} = N \int [dA] \Delta_F[A] \exp\left\{iS[A] - \frac{i}{2\alpha} \int d^4x (F^a[A])^2\right\}. \tag{2.20}$$

Eq. (2.20) is the starting point of our entire discussion. We define the generating functional  $W_F[J_\mu^a]$  of the Green functions in the gauge specified by  $F$  to be

$$\begin{aligned} \exp\{iW_F[J_\mu^a]\} &= N \int [dA] \Delta_F[A] \\ &\times \exp\left\{i \int d^4x \left[\mathcal{L}(x) - \frac{1}{2\alpha} F_a^2 - J^{\mu a} A_\mu^a\right]\right\}. \end{aligned} \tag{2.21}$$

Please note that the above is a definition. We have not answered yet how Green functions in different gauges are related to each other, or to the physical S-matrix. We shall return to these questions in a future lecture.

2.2. Faddeev-Popov ghosts

As we have noted in the preceding section,  $\Delta_F[A]$  has the structure of a determinant. Such a determinant occurs frequently in path integrals.

Consider a complex scalar field  $\varphi$  interacting with a prescribed external potential  $V(x)$ . The vacuum-to-vacuum amplitude is written

$$e^{iW} = N \int [d\varphi][d\varphi^\dagger] \exp\{i \int d^4x \varphi^\dagger(x) [-\partial^2 - \mu^2 + V(x)] \varphi(x)\} \sim (\det M(x, y))^{-1}, \tag{2.22}$$

where

$$M(x, y) = [-\partial^2 - \mu^2 + V(x)] \delta^4(x - y). \tag{2.23}$$

On the other hand, we can evaluate  $W$  in perturbation theory: it is a sum of vacuum loop diagrams shown in the following figure:

$$W = V \text{ (circle) } + V \text{ (figure-eight) } + V \text{ (circle with V on top) } + \dots$$

This result can be understood in the following way. We write

$$(\det M(x, y))^{-1} = (\det M_0(x, y))^{-1} \times (\det[\delta^4(x - y) + \Delta_F(x - y)V(y)])^{-1}, \tag{2.24}$$

where

$$M_0(x, y) = (-\partial^2 - \mu^2) \delta^4(x - y), \tag{2.25}$$

$$\Delta_F(x - y) = \left\langle x \left| \frac{1}{-\partial^2 - \mu^2 + i\epsilon} \right| y \right\rangle. \tag{2.26}$$

(The  $i\epsilon$  prescription in eq. (2.26) follows from the Euclidicity postulate inherent in the definition of path integrals. See refs. [1,2]). The first factor on the right-hand side of eq. (2.24) may be absorbed in the normalizing factor  $N$ . The second factor may be evaluated with the aid of the formula

$$\det(1 + L) = \exp \text{Tr} \ln(1 + L).$$

Thus,

$$iW = -\text{Tr} \ln(1 + \Delta_F V), \tag{2.27}$$

which shows very clearly  $W$  as a sum of loops.

Next, what if  $\varphi$  and  $\varphi^\dagger$  were anticommuting fields? Nothing much changes, except that each closed loop acquires a minus sign. Thus, if  $\eta$  and  $\eta^\dagger$  are anticommuting fields, we have

$$\begin{aligned} e^{iW} &= N \int [d\eta] [d\eta^\dagger] \exp \left\{ i \int d^4x \eta^\dagger(x) [-\partial^2 - \mu^2 + V(x)] \eta(x) \right\} \\ &\sim \det M(x, y) \sim \exp \{ \text{Tr} \ln(1 + \Delta_F V) \}. \end{aligned}$$

The above is a heuristic argument of how integrals over anticommuting  $c$ -numbers should be defined to be useful in the formulation of field theory. In fact, Berezin defines the integral over an element of Grassmann algebra  $c_i$  as

$$\int dc_i = 0, \quad \int dc_i c_j = \delta_{ij}.$$

It then follows

$$\int \prod_i dc_i e^{c_i A_{ij} c_j} \sim (\det A)^{1/2}. \tag{2.28}$$

(*Problem:* prove this statement.) We shall not dwell upon the integration over Grassmann algebra any further, but rather refer you to Berezin's treatise to be cited at the end of this lecture. A nice mnemonic for the rules of integration over anticommuting  $c$ -numbers is, as told to me by Jean Zinn-Justin, that "integration is equivalent to derivation".

For our purpose, we can write

$$\Delta_F[A] \sim \det M_F = N \int [d\xi] [d\eta] \exp \left\{ i \int d^4x \xi_a \frac{\partial F^a}{\partial A_\mu^c} D_\mu^{cb} \eta_b \right\}, \tag{2.29}$$

or symbolically

$$\Delta_F[A] = N \int [d\xi][d\eta] \exp\{i\xi M_F \eta\}, \quad (2.30)$$

where  $\xi_a(x)$ ,  $\eta_b(x)$  are elements of Grassmann algebra.

Note that the phase of the exponent  $\xi M_F \eta$  is purely conventional.

The generating functional  $W_F[J_\mu^a]$  of eq. (2.21) can now be written as

$$\begin{aligned} \exp\{iW_F[J_\mu^a]\} &= N \int [dA d\xi d\eta] \\ &\times \exp\{iS_{\text{eff}}[A, \xi, \eta] - i \int d^4x J^{\mu a}(x) A_\mu^a(x)\}, \end{aligned} \quad (2.31)$$

where the effective action  $S_{\text{eff}}$  is given by

$$S_{\text{eff}}[A, \xi, \eta] = \int d^4x \left[ \mathcal{L}(x) - \frac{1}{2\alpha} (F^a[A(x)])^2 + \xi_a \frac{\partial F^a}{\partial A_\mu^c} D_\mu^{cb} \eta_b(x) \right]. \quad (2.32)$$

The fields  $\xi$ ,  $\eta$  are usually called the Faddeev-Popov ghost fields. They are unphysical scalar fields which anticommute among themselves. (Sometimes it is convenient to think of  $\xi$  as hermitian conjugate of  $\eta$ , but it is not necessary.)

In the Lorentz gauge, where we shall write

$$F^a = -\partial^\mu A_\mu^a,$$

the term in the effective action bilinear in  $\xi$  and  $\eta$  is

$$\begin{aligned} \int d^4x \xi_a \frac{\partial F^a}{\partial A_\mu^c} D_\mu^{cb} \eta_b(x) &= \int d^4x \partial^\mu \xi_a(x) D_\mu^{ab} \eta_b(x) \\ &= \int d^4x [\partial^\mu \xi_a(x) \partial_\mu \eta_a(x) - g \partial^\mu \xi_a(x) F_{abc} A_\mu^c(x) \eta_b(x)]. \end{aligned} \quad (2.33)$$

Even when we regard  $\xi$  and  $\eta$  as a conjugate pair, the interaction of eq. (2.33) is not hermitian. The sole raison d'être of this term is to create the determinantal factor, eq. (2.29).

### 2.3. Feynman rules

To describe the Feynman rules for constructing Green functions in perturbation theory, it is more convenient to couple  $\xi_a$  and  $\eta_a$  also to their own sources  $\beta_a$  and  $\beta_a^\dagger$ , which are anticommuting  $c$ -numbers. We define

$$\exp\{iW_F[J_\mu, \beta, \beta^\dagger]\} = N \int [dA][d\xi][d\eta] \quad (2.34)$$

$$\times \exp\left[i \int d^4x \left\{ \mathcal{L}(x) - \frac{1}{2\alpha} F^2 + \xi \cdot \beta + \beta^\dagger \cdot \eta - J_\mu \cdot A^\mu + \xi \cdot \frac{\partial F}{\partial A_\mu} D_\mu \eta \right\}\right],$$

where we have suppressed gauge group indices.

The Feynman rules are obtained from eq. (2.34) in the usual way. We will review briefly the derivation of the Feynman rules in a simpler example, an interacting real scalar field  $\varphi$ . The action  $S[\varphi]$  is divided into two parts,

$$S[\varphi] = S_0[\varphi] + S_I[\varphi], \quad (2.35)$$

where  $S_0[\varphi]$  is the part quadratic in the field  $\varphi$ , and has the form

$$S_0[\varphi] = \int d^4x \left[ \frac{1}{2} (\partial_\mu \varphi)^2 - \frac{1}{2} \mu^2 \varphi^2 \right]. \quad (2.36)$$

The generating functional  $iW[J]$  of the connected Green functions is given by

$$\begin{aligned} e^{iW[J]} &= \int [d\varphi] \exp\{iS[\varphi] + i \int d^4x J\varphi\} \\ &= \exp\left\{iS_I\left[-i \frac{\delta}{\delta J}\right]\right\}_N \int [d\varphi] \exp\{iS_0[\varphi] + i \int d^4x J\varphi\}. \end{aligned} \quad (2.37)$$

Therefore, we must now compute

$$e^{iW_0[J]} = N \int [d\varphi] \exp\{iS_0[\varphi] + i \int d^4x J\varphi\}. \quad (2.38)$$

Since  $S_0$  is quadratic in  $\varphi$ , we can perform the integration.

The functional integral in eq. (2.38) gains a well-defined meaning by the Euclidicity postulate, that the Green functions eq. (2.37) generates must be the analytic continuations of the well-behaved Euclidean Green functions.

We obtain

$$e^{iW_0[J]} = \exp\left\{-\frac{1}{2} i \int d^4x d^4y J(x) \Delta_F(x-y) J(y)\right\}, \quad (2.39)$$

where

$$\Delta_F(x) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik \cdot x}}{k^2 - \mu^2 + i\epsilon}. \quad (2.40)$$

Eq. (2.37), or

$$\exp\{iW[J]\} = \exp\left\{iS_J\left[\frac{1}{i}\frac{\delta}{\delta J}\right]\right\} \exp\{iW_0[J]\}, \quad (2.37')$$

may be transformed into a perhaps more tractable form by the use of the formula

$$F\left(-i\frac{\partial}{\partial x}\right)G(x) = G\left(-i\frac{\partial}{\partial y}\right)F(y)e^{ix \cdot y}\Big|_{y=0},$$

which can be proved by Fourier analysis (see ref. [2]),

$e^{iW[J]}$

$$\begin{aligned} e^{iW[J]} &= \exp\left\{\frac{1}{2}i\int d^4x d^4y \Delta_F(x-y) \frac{\delta}{\delta \varphi(x)} \frac{\delta}{\delta \varphi(y)}\right\} \\ &\times \exp\{iS_1[\varphi] + i\int d^4x J\varphi\}\Big|_{\varphi=0}. \end{aligned} \quad (2.41)$$

In much the same way, we can develop the Feynman rules for the gauge theory from eq. (2.34). To be concrete, let us adopt the Lorentz gauge,  $F_a = -\partial^\mu A_\mu^a$ . We define  $S_0$  to be

$$\begin{aligned} S_0[A, \xi, \eta] &= \int d^4x \left[ -\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 - \frac{1}{2\alpha}(\partial_\mu A_\mu^a)^2 \right. \\ &\left. + \partial^\mu \xi^a \partial_\mu \eta_a + \xi \cdot \beta + \beta^\dagger \cdot \eta - J^\mu \cdot A_\mu \right]. \end{aligned} \quad (2.42)$$

The remainder of the action  $S$  consists of the cubic and quartic interactions of the gauge fields and the interaction of the gauge field with the ghost fields. The Feynman propagator for the gauge bosons satisfies

$$\left[ \partial^2 g_{\mu\nu} - \partial_\mu \partial_\nu \left(1 - \frac{1}{\alpha}\right) \right] \Delta_F^{\nu\lambda}(x-y) = g_\mu^\lambda \delta^4(x-y), \quad (2.43)$$

and is given by

$$\Delta_F^{\mu\nu}(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot (x-y)} \left[ -\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) + \alpha \frac{k^\mu k^\nu}{k^2} \right] \frac{1}{k^2 + i\epsilon}. \quad (2.44)$$

Note that in this gauge the ghost field  $\xi^a$  always appears as  $\partial_\mu \xi^a$ . The generating functional  $W_L[J_\mu, \beta, \beta^\dagger]$  can be written as

$$\begin{aligned}
\exp\{iW_L[J_\mu, \beta, \beta^\dagger]\} &= \exp\left\{i \int d^4x d^4y \left[ \Delta_F^{\mu\nu}(x-y) \frac{\delta}{\delta A^{\mu a}(x)} \frac{\delta}{\delta A^{\nu a}(y)} \right. \right. \\
&\quad \left. \left. + D_F(x-y) \frac{\delta}{\delta \eta(x)} \frac{\delta}{\delta \xi(y)} \right] \right\} \exp(i\{S_I[A_\mu, \xi, \eta] \\
&\quad + \int d^4x [\xi \cdot \beta + \beta^\dagger \cdot \eta + J^\mu \cdot A_\mu](x)\}) \Big|_{A_\mu = \xi = \eta = 0}, \quad (2.45)
\end{aligned}$$

where  $D_F(x-y)$  is the Feynman propagator for a massless scalar field,

$$D_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik \cdot (x-y)}}{k^2 + i\epsilon}.$$

That is, in this gauge, the Faddeev-Popov ghosts are massless.

#### 2.4. Mixed transformations

A few remarks on the integration over elements of the Grassmann algebra: Since we wish to maintain the integration rule

$$\int \prod_{i=1}^n dc_i \prod_{j=1}^n c_j = 1$$

under a change of variables,

$$c_i = A_{ij} \xi_j, \quad \prod_{i=1}^n c_i = \prod_{j=1}^n \xi_j \det A,$$

we must have

$$\prod_i dc_i = (\det A)^{-1} \prod_i d\xi_i.$$

Further, let us consider a mixed multiple integral of the form

$$\int \prod_i dx_i \prod_\mu d\theta_\mu,$$

where  $\theta$ 's are elements of the Grassmann algebra. We consider a change of integration variables of the form

$$\begin{pmatrix} x \\ \theta \end{pmatrix} = \begin{pmatrix} A & \alpha \\ \beta & B \end{pmatrix} \begin{pmatrix} y \\ \varphi \end{pmatrix}$$

and ask how the Jacobian must be defined. We consider first the change  $(x, \theta) \rightarrow (y, \theta)$ ,

$$\begin{aligned} x &= Ay + \alpha B^{-1}\theta - \alpha B^{-1}\beta y \\ &= (A - \alpha B^{-1}\beta)y + \alpha B^{-1}\theta. \end{aligned}$$

Note that  $\alpha$  and  $\beta$  are anticommuting. Thus,

$$\int \prod dx_i \prod d\theta_\mu = \int \prod d\theta_\mu \int \prod dy_i \det(A - \alpha B^{-1}\beta).$$

Now we perform the transformation  $(y, \theta) \rightarrow (y, \varphi)$ ,

$$\theta = \beta y + B\varphi.$$

Since  $\theta$  and  $\varphi$  are anticommuting numbers,

$$\int \prod d\theta_\mu = \int \prod d\varphi_\mu (\det B)^{-1}.$$

As a result, we have the rule

$$\int \prod dx_i \prod d\theta_\mu = \int \prod dy_i \prod d\varphi_\mu \det(A - \alpha B^{-1}\beta)(\det B)^{-1}.$$

The above result is in accord with the definition of a "generalized determinant" or Arnowitt, Nath and Zumino. They define the determinant of the matrix

$$C = \begin{pmatrix} A & \alpha \\ \beta & B \end{pmatrix}$$

by

$$\det C = \exp \text{Tr} \ln C,$$

with the convention that

$$\text{Tr} C = C_{ii} - C_{\mu\mu}.$$

This definition allows the following relation to be valid:

$$\text{Tr } C_1 C_2 = \text{Tr } C_2 C_1 ,$$

and therefore the product property of the determinant

$$\det(C_1 C_2) = \det C_1 \det C_2 .$$

To see this, we set  $C_i = \exp J_i$ , so that  $\det C_i = \exp \text{Tr } J_i$ . Now  $C_1 C_2 = \exp \{J_1 + J_2 + J_{12}\}$  where  $J_{12}$  is the Baker-Hansdorff series of commutators. But  $\text{Tr}[J_1, J_2] = 0$ , etc., so that  $\det(C_1 C_2) = \exp \text{Tr}(J_1 + J_2)$ . Now the matrix  $C$  can be decomposed uniquely into the form  $ST$ , where

$$S = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 \\ y & z \end{pmatrix},$$

and

$$\begin{aligned} a &= A - \alpha B^{-1} \beta, & b &= \alpha B^{-1}, \\ y &= \beta, & z &= B. \end{aligned}$$

Thus

$$\det C = \det S \det T = (\det a)(\det z)^{-1} .$$

(The last follows from the definition  $\det C = \exp[(\ln C)_{ii} - (\ln C)_{\mu\mu}]$ .)

### 2.5. Problems

2.5.1. One should repeat the foregoing arguments for quantum electrodynamics, to obtain the usual Feynman rules in the Lorentz gauge. Let us note that for  $\alpha = 0$ , one gets the photon propagator in the Landau gauge; for  $\alpha = 1$ , that in the Feynman gauge. What happens to the Faddeev-Popov ghost fields in those cases?

2.5.2. Just for the sake of exercise, quantize electrodynamics in the gauge  $F = \partial^\mu A_\mu + \lambda A_\mu^2$ . Derive the Feynman rules.

2.5.3. Show that  $\int \prod_i dc_i \prod_j dc'_j \exp \{c_i M_{ij} c'_j\} \sim \det M$ , where  $c$  and  $c'$  are anticommuting.

### 3. Survey of renormalization schemes

#### 3.1. Necessity for a gauge-invariant regularization

In this lecture, we will develop two subjects that are needed to understand later lectures. These are regularization and renormalization of Green functions in quantum field theory in general, and of Green functions in a gauge theory generated by the expression (2.34), in particular.

The Green functions generated by eq. (2.34) are plagued by the ultraviolet infinities encountered in any realistic quantum field theory. We are going to develop a method of eliminating these divergences by redefinitions, or renormalizations of basic parameters and fields in the theory, in such a way that the gauge invariance of the original Lagrangian is unaffected in so doing.

The gauge invariance of the action has various implications on the structure of Green functions of the theory. The precise mathematical expressions which are satisfied by Green functions due to the gauge symmetry of the underlying action are known as the Ward-Takahashi (WT) identities. What we will show is that these identities remain form invariant under renormalization which eliminate the divergences. This point, that renormalization can be carried out in a way that preserves the WT identities, is of utmost importance for the following reasons. First, it puts such a stringent constraint on the theory and the renormalization procedure that the renormalized theory becomes unique, once the underlying renormalizable theory is given. Second, and perhaps more to the point, the unitarity of the renormalized  $S$ -matrix is shown by the WT identities satisfied by the renormalized Green functions. The latter point requires clarification.

In a perturbative approach, non-Abelian gauge theories suffer from such severe infrared singularities that nobody has succeeded in defining a sensible  $S$ -matrix in this framework. Consensus is that a sensible gauge theory arises only in a non-perturbative approach, wherein gauge fields and other matter fields carrying non-Abelian charges do not manifest themselves as physical particles. Physically, this conjecture is at the heart of the hope that color-quark confinement might arise naturally from a non-Abelian gauge theory of strong interactions. There is an exception to this, and this is the case when the gauge symmetry is spontaneously broken. In fact, this latter possibility is directly responsible for the revival of interest in non-Abelian gauge theories a few years ago, in conjunction with efforts to unify electromagnetic and weak interactions in a non-Abelian gauge theory. In this case there is no difficulty in defining the physical  $S$ -matrix, and the unitarity of such a theory is assured by the renormalized version of the WT identities. Even in unbroken gauge theory,

the  $S$ -matrix can be defined up to some lower order in perturbation theory, and here again the unitarity of the  $S$ -matrix is a consequence of the WT identities.

Why is the unitarity such a big issue in gauge theory? After all, one does not worry that much about the unitarity, say, in a self-interacting scalar boson theory. The reason is that the quantization procedure we adopted makes use of a non-positive definite Hilbert space, as we can readily see from the structure of the gauge boson propagator, eq. (2.44). Further, the Green functions of the theory contain singularities arising from the Faddeev-Popov ghosts being on the mass shell. Thus, in order that the theory makes sense, these unphysical "particles", corresponding to the ghost fields and the longitudinal components of gauge fields, must decouple from the physical  $S$ -matrix. The renormalized WT identities are necessary in showing this.

The WT identities are usually derived by a formal manipulation of eq. (2.34). However, the Green functions generated by eq. (2.34) are notoriously ill-defined objects due to ultraviolet divergences. It is therefore necessary to invent a means of "regularizing" the Feynman integrals which define them without destroying symmetry properties of the Green functions, so that as long as we keep a regularization parameter finite, the integrals are well-defined. It is only then that we can attach concrete meaning to the WT identities. After renormalization, the "regulator" may be removed, and if the renormalization is to be successful, the renormalized Green functions must be finite and independent of the regularization parameter.

A well-known regularization scheme in quantum electrodynamics is the Pauli-Villars scheme, in which one adds unphysical fields of variable masses to the Lagrangian in a gauge invariant way. After gauge invariant renormalization the variable masses are let go to infinity, and renormalized quantities are shown to be finite in this limit. In non-Abelian gauge theory, this device is not available, but an alternative procedure, wherein the dimensionality of space-time is continuously varied, was invented by the genius of 't Hooft and Veltman.

In the next section, we will give a brief summary of the renormalization theory à la Bogoliubov, Parasiuk, Hepp and Zimmermann. This will be followed by an introduction to the dimensional regularization of 't Hooft and Veltman.

### 3.2. BPHZ renormalization

In this section we will give a brief survey of renormalization theory developed and perfected in recent years by Bogoliubov, Parasiuk, Hepp and Zimmermann (BPHZ). Nothing will be proved, but we will try to give definitions and theorems in a precise manner.

First, we will give some definitions. The interaction Lagrangian is a sum of terms  $\mathcal{L}_i$  which is a product of  $b_i$  boson fields and  $f_i$  fermion fields with  $d_i$  derivatives. The vertex of the  $i$ th type arising from  $\mathcal{L}_i$  has the index  $\delta_i$  defined as

$$\delta_i = b_i + \frac{3}{2}f_i + d_i - 4 = \dim(\mathcal{L}_i) - 4. \quad (3.1)$$

Let  $\Gamma$  be a one-particle irreducible (IPI) diagram (i.e., a diagram that cannot be made disconnected by cutting only one line). Let  $E_B$  and  $E_F$  be the numbers of external boson and fermion lines,  $I_B$  and  $I_F$  the numbers of internal boson and fermion lines,  $n_i$  the number of vertices of the  $i$ th type. Then

$$E_B + 2I_B = \sum_i n_i b_i, \quad (3.2)$$

$$E_F + 2I_F = \sum_i n_i f_i. \quad (3.3)$$

The superficial degree of divergence of  $\Gamma$  is the degree of divergence one would naively guess by counting the powers of momenta in the numerator and denominator of the Feynman integral. It is

$$D(\Gamma) = \sum_i n_i d_i + 2I_B + 3I_F - 4V + 4, \quad (3.4)$$

the last two terms arising from the fact that at each vertex there is a four-dimensional delta function which allows one to express one four-momentum in terms of other momenta, except that one delta function expresses the conservation of external momenta. Making use of eqs. (3.1), (3.2) and (3.3), we can write eq. (3.4) as

$$D = \sum_i n_i \delta_i - E_B - \frac{3}{2}E_F + 4, \quad (3.5)$$

or

$$D + E_B + \frac{3}{2}E_F - 4 = \sum_i n_i \delta_i. \quad (3.6)$$

The purpose of renormalization theory is to give a definition of the finite part of the Feynman integral corresponding to  $\Gamma$ ,

$$F_\Gamma = \lim_{\epsilon \rightarrow 0^+} \int dk_1 \dots dk_L I_\Gamma, \quad (3.7)$$

where  $I_\Gamma$  is a product of propagators  $\Delta_F$  and vertices  $P$ ,

$$I_\Gamma = \prod_{a,b,\sigma} \Delta_F^{ab\sigma} \prod_a P_a. \quad (3.8)$$

The finite part of  $F_\Gamma$  will be denoted by  $J_\Gamma$  and written

$$J_\Gamma = \lim_{\epsilon \rightarrow 0_+} \int dk_1 \dots dk_L R_\Gamma. \quad (3.9)$$

We shall describe Bogoliubov's prescription of constructing  $R_\Gamma$  from  $I_\Gamma$ .

Let us first consider a simple case, in which  $\Gamma$  is primitively divergent. The diagram  $\Gamma$  is primitively divergent if it is proper (i.e., IPI), superficially divergent (i.e.,  $D(\Gamma) \geq 0$ ) and becomes convergent if any line is broken up. In this case, we may use the original prescription of Dyson. We write

$$J_\Gamma = \int dk_1 \dots dk_L (1 - t^\Gamma) I_\Gamma,$$

i.e.,

$$R_\Gamma = (1 - t^\Gamma) I_\Gamma.$$

The operation  $t^\Gamma$  must be defined to cancel the infinity in  $J_\Gamma$ .  $J_\Gamma$  is a function of  $E_F + E_B - 1 = E - 1$  external momenta  $p_1, \dots, p_{E-1}$ ,

$$I_\Gamma = f(p_1, \dots, p_{E-1}).$$

The operation  $(1 - t^\Gamma)$  on  $f$  is defined by subtracting from  $f$  the first  $D(\Gamma) + 1$  terms in a Taylor expansion about  $p_i = 0$ ,

$$t^\Gamma f(p_1, \dots, p_{E-1}) = f(0, \dots, 0) + \dots \quad (3.10)$$

$$+ \frac{1}{d!} \sum_{i_1, \dots, i_d=1}^{E-1} (p_{i_1})_\lambda (p_{i_2})_\mu \dots (p_{i_d})_\nu \frac{\partial^d f}{(\partial p_{i_1})_\lambda (\partial p_{i_2})_\mu \dots (\partial p_{i_d})_\nu},$$

where  $d = D(\Gamma)$ . The operation  $(1 - t^\Gamma)$  amounts to making subtractions in the integrand  $I_\Gamma$ , the number of subtractions being determined by the superficial degree of divergence of the integral.

Some more definitions: A renormalization part is a proper diagram which is superficially divergent ( $D \geq 0$ ). Two diagrams (subdiagrams) are disjoint,  $\gamma_1 \cap \gamma_2 \neq \phi$  if they have no lines or vertices in common. Let  $\{\gamma_1, \dots, \gamma_c\}$  be a set of mutually disjoint connected subdiagrams of  $\Gamma$ . Then

$$F \equiv \Gamma / \{\gamma_1, \dots, \gamma_c\}$$

is defined by contracting each  $\gamma$  to a point and assigning the value 1 to the corresponding vertex.

We are now in a position to describe Bogoliubov's  $R$  operation:

(i) if  $\Gamma$  is not a renormalization part (i.e.,  $D(\Gamma) \leq -1$ ),

$$R_\Gamma = \bar{R}_\Gamma, \quad (3.11)$$

(ii) if  $\Gamma$  is a renormalization part ( $D(\Gamma) \geq 0$ ),

$$R_\Gamma = (1 - t^\Gamma) \bar{R}_\Gamma, \quad (3.12)$$

where  $\bar{R}_\Gamma$  is defined as

$$\bar{R}_\Gamma = I_\Gamma + \sum_{\{\gamma_1, \dots, \gamma_c\}} I_{\Gamma/\{\gamma_1, \dots, \gamma_c\}} \prod_{\tau=1}^c O_{\gamma_\tau}, \quad (3.13)$$

and  $O_\gamma = -t^\gamma \bar{R}_\gamma$ , where the sum is over all possible different sets of  $\{\gamma_i\}$ . This definition of  $\bar{R}_\Gamma$  in terms of  $\bar{R}_\gamma$  appears to be recursive; in perturbation theory there is no problem; the  $\bar{R}_\gamma$  appearing in the definition of  $\bar{R}_\Gamma$  is necessarily of lower order.

It is possible to "solve" eq. (3.13). We refer the interested reader to Zimmermann's lectures and merely present the result. Again we need some more definitions before we can do this. Two diagrams  $\gamma_1$  and  $\gamma_2$  are said to overlap,  $\gamma_1 \cap \gamma_2$ , if none of the following holds:

$$\gamma_1 \cap \gamma_2 = \phi, \quad \gamma_1 \supset \gamma_2, \quad \gamma_2 \supset \gamma_1.$$

A  $\Gamma$ -forest  $U$  is a hierarchy of subdiagrams satisfying (a)–(c) below:

(a) elements of  $U$  are renormalization parts; (b) any two elements of  $U$ ,  $\gamma'$  and  $\gamma''$  are non-overlapping; (c)  $U$  may be empty. A  $\Gamma$ -forest  $U$  is full or normal respectively depending on whether  $U$  contains  $\Gamma$  itself or not. The theorem due to Zimmermann is

$$R_\Gamma = \sum_{\text{all } U} \prod_{\lambda \in U} (-t^\lambda) I_\Gamma, \quad (3.14)$$

where  $\Sigma$  extends over all possible (full, normal and empty)  $\Gamma$  forests, and in the product  $\prod(-t^\lambda)$  the factors are ordered such that  $t^\lambda$  stands to the left of  $t^\sigma$  if  $\lambda \supset \sigma$ . If  $\lambda \cap \sigma = \phi$ , the order is irrelevant. A simple example is in order. Consider the diagram in fig. 3.1. The forests are  $\phi$  (empty);  $\gamma_1$  (full);  $\gamma_2$  (normal);  $\gamma_1, \gamma_2$  (full). Eq. (3.14) can be written in this case as

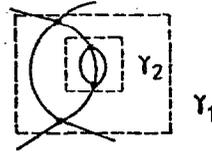


Fig. 3.1. Example of the BPHZ definition of subdiagrams in a particular contribution to the four-point function in a  $\lambda\Phi^4$  coupling theory.

$$R_\Gamma = (1 - t^{\gamma_1} - t^{\gamma_2} + t^{\gamma_1} t^{\gamma_2}) I_\Gamma = (1 - t^{\gamma_1})(1 - t^{\gamma_2}) I_\Gamma .$$

Note that in the BPH program, the  $R$ -operation is performed with respect to subdiagrams which consists of vertices and all propagators in  $\Gamma$  which connect these vertices. By the BPH definition, the subdiagram  $\gamma_2$  above does not contain renormalization parts other than itself and in this sense the present treatment differs from Salam's discussion.

In formulating the BPH theorem it is necessary first to regularize the propagators in eq. (3.9) by some device such as

$$\Delta_F(p) \xrightarrow{\text{reg}} \Delta_F(p; r, \epsilon) = -i \int_r^\infty d\alpha \exp[i\alpha(p^2 - m^2 + i\epsilon)] ,$$

and define  $I_\Gamma(r, \epsilon)$  as in eq. (3.9) in terms of  $\Delta_F(r, \epsilon)$ , and then construct  $R_\Gamma(r, \epsilon)$  by the  $R$ -operation. The BPH theorem states that  $R_\Gamma$  exists as  $r \rightarrow 0$  and  $\epsilon \rightarrow 0^+$ , as a boundary value of an analytic function in the external momenta. Another theorem, the proof of which can be found in the book by Bogoliubov and Shirkov, sect. 26, and which is combinatoric in nature, states that the subtractions implied by the  $(1 - t^\Gamma)$  prescription in the  $R$ -operation can be formally implemented by adding counterterms in the Lagrangian.

A theory which has a finite number of renormalization parts is called renormalizable. A theory in which all  $\delta_i$  are less than, or equal to zero is renormalizable. In this case the index of a subtraction term in the  $R$ -operation is bounded by  $D + E_B + \frac{3}{2}E_F - 4$  which is at most equal to zero by eq. (3.5). In such a theory, only a finite number of renormalization counterterms to the Lagrangian suffice to implement the  $R$ -operation.

### 3.3. The regularization scheme of 't Hooft and Veltman

Recently, 't Hooft and Veltman proposed a scheme for regularizing Feyn-

man integrals which preserves various symmetries of the underlying Lagrangian. This method is applicable to electrodynamics, and non-Abelian gauge theories, and depends on the idea of analytic continuation of Feynman integrals in the number of space-time dimensions. The critical observations here are that the global or local symmetries of these theories are independent of space-time dimensions, and that Feynman integrals are convergent for sufficiently small, or complex  $N$ , where  $N$  is the "complex dimension" of space-time.

Let us first review the nature of ultraviolet divergence of a Feynman diagram. For this purpose, it is convenient to parametrize the propagators as

$$\Delta_F(p^2) = \frac{1}{i} \int_0^\infty d\alpha \exp[i\alpha(p^2 - m^2 + i\epsilon)]. \quad (3.15)$$

Making use of this representation, we can write a typical Feynman integral as

$$F_\Gamma \sim \left( \prod_{i=1}^I \int_0^\infty d\alpha_i \right) \left( \prod_{j=1}^L \int d^4 k_j \right) (k_{l_1})_\lambda (k_{l_2})_\mu \dots (k_{l_n})_\nu \\ \times \exp\{i \sum_i \alpha_i (q_i^2 - m_i^2 + i\epsilon)\}, \quad (3.16)$$

where  $I$  is the number of internal propagators in  $\Gamma$ ,  $L$  the number of loops, and  $l_1, \dots, l_n$  may take any values from 1 to  $L$ . The momentum  $q_j$  carried by the  $j$ th propagator is a linear function of loop momenta  $k_i$  and external momenta  $p_m$ . The exponent on the right-hand side of eq. (3.16) can therefore be written as

$$\sum_{i=1}^I \alpha_i (q_i^2 - m_i^2 + i\epsilon) = \frac{1}{2} \sum_{i,j} k_i A_{ij}(\alpha) k_j + \sum_{i,m} k_i B_{im}(\alpha) p_m - \sum_i \alpha_i (m_i^2 - i\epsilon) \\ \equiv \frac{1}{2} k^T \cdot A \cdot k + k \cdot B \cdot p - \sum_i \alpha_i (m_i^2 - i\epsilon),$$

where  $k$  is a column matrix with entries which are four-vectors. The matrices  $A$  and  $B$  are homogeneous functions of first degree in  $\alpha$ 's, and  $A$  is symmetric. Upon translating the integration variables

$$k \rightarrow k' = k + A^{-1} B p$$

and diagonalizing the matrix  $A$  by an orthonormal transformation on  $k'$ , we can perform the loop integrations over  $k_j$  in eq. (3.16). The result is a sum over terms each of which has the form

$$F_{\Gamma} \sim T_{\lambda\mu \dots \nu} \left( \prod_{i=1}^I \int_0^{\infty} d\alpha_i \right) \frac{1}{\prod_i [A_i(\alpha)]^{s_i}} \times \exp \left\{ -i \left[ \frac{1}{2} p \cdot C(\alpha) \cdot p + \sum_i \alpha_i (m_i^2 - i\epsilon) \right] \right\}, \quad (3.17)$$

where  $T_{\lambda\mu \dots \nu}$  is a tensor, typically a product of  $g_{\rho\sigma}$ 's,  $A_i(\alpha)$  is the  $i$ th eigenvalue of the matrix  $A$ , and  $s_i$  is a positive number which is determined by the tensorial structure of  $F_{\Gamma}$ . Note that  $A_i(\alpha)$  is homogeneous of first degree in  $\alpha$ 's. The matrix  $C$  is

$$C = B^T A^{-1} B$$

and is also a homogeneous function of first degree in  $\alpha$ 's. In this parametrization, the ultraviolet divergences of the integral appear as the singularities of the integrand on the right-hand side of eq. (3.17) arising from the vanishing of some factors  $\Pi_i [A_i(\alpha)]^{s_i}$  as some or all  $\alpha$ 's approach to zero in certain orders, for example,

$$\alpha_{r_1} < \alpha_{r_2} < \dots < \alpha_{r_I},$$

where  $(r_1, r_2, \dots, r_I)$  is a permutation of  $(1, 2, \dots, I)$ . See, for instance, a more detailed and careful discussion of Hepp.

The 't Hooft-Veltman regularization consists in defining the integral  $F_{\Gamma}$  in  $n$  dimensions,  $n > 4$  (one-time and  $(n - 1)$ -space dimensions) while keeping external momenta and polarization vectors in the first four dimensions (i.e., in the physical space), performing the  $n - 4$  dimensional integrals in the space orthogonal to the physical space, and then continuing the result in  $n$ . (For single-loop graphs one may perform all  $n$  integrations together.) For sufficiently small  $n$ , or complex  $n$ , the subsequent four-dimensional integrations are convergent.

To see how it works, consider the integral

$$F_{\Gamma}(n) \sim \left( \prod_i \int d\alpha_i \right) \left( \prod_{j=1}^L \int d^n k_j \right) \prod (k_a \cdot k_b) \prod (k_c \cdot p_m) \\ \times \prod (k_d \cdot e_j) \exp[i \sum_i \alpha_i (q_i^2 - m_i^2 + i\epsilon)], \quad (3.18)$$

where, now, the  $k_j$  are  $n$ -dimensional vectors. As before we can express the  $q_i$  as linear functions of the  $k_i$  and the external momenta  $p_i$ , where the  $p_i$  have only first four-component non-vanishing. From now, we shall denote an  $n$ -dimensional vector by  $(\hat{k}, K)$  where  $\hat{k}$  is the projection of  $k$  onto the physical space-time and  $K = k - \hat{k}$ . Thus,  $p = (\hat{p}, 0)$ . Eq. (3.18) may be written as a sum of terms of the form

$$F_{\Gamma}(n) \sim \left( \prod_{i=1}^I \int d\alpha_i \right) \left( \prod_{j=1}^L \int d^4 \hat{k}_j \right) \left( \prod_{j=1}^L \int d^{n-4} K_j \right) \left( \prod_{a,b} K_a \cdot K_b \right) \\ \times \left( \prod_{c,m} \hat{k}_c \cdot \hat{p}_m \right) \left( \prod_{d,l} \hat{k}_d \cdot \hat{e}_l \right) \left( \prod_{e,f} \hat{k}_e \cdot \hat{k}_f \right) \quad (3.19) \\ \times \exp[i\{\hat{k}^T \cdot \frac{1}{2} A \hat{k} + \hat{k} \cdot B p - K^T \cdot \frac{1}{2} A K - i \sum \alpha_i (m_i^2 - i\epsilon)\}].$$

The integrals over  $K_j$  can be performed immediately, using the formulas

$$\int d^{n-4} K K_{\alpha_1} K_{\alpha_2} \dots K_{\alpha_{2r}} \exp(-i A K^2) \\ = \frac{\pi^{n/2}}{2^r r!} \sum_{\sigma \in S_{2r}} \delta_{\sigma(\alpha_1), \sigma(\alpha_2)} \delta_{\sigma(\alpha_3), \sigma(\alpha_4)} \dots \delta_{\sigma(\alpha_{2r-1}), \sigma(\alpha_{2r})} \\ \times (iA)^{-n/2+2-r},$$

where the summation is over the elements  $\sigma$  of the symmetric group on  $2r$  objects  $(\alpha_1, \alpha_2, \dots, \alpha_{2r})$ , and

$$\delta_{\alpha\beta} \delta_{\beta\alpha} = n - 4.$$

Thus  $F_{\Gamma}$  of eq. (3.19) will have the form

$$F_{\Gamma}(n) \sim \left( \prod_i \int d\alpha_i \right) \frac{f(n)}{\prod_i [A_i(\alpha)]^{n/2-2+r_i}} \left( \prod_j \int d^4 \hat{k}_j \right) \prod_{e,f} (\hat{k}_e \cdot \hat{k}_f) \\ \times \prod_{c,m} (\hat{k}_c \cdot \hat{p}_m) \prod_{d,l} (\hat{k}_d \cdot \hat{e}_l) \exp[i \prod_i \alpha_i (\hat{q}_i^2 - m_i^2 + i\epsilon)],$$

where  $f(n)$  is a polynomial in  $n$  and  $r_i$  is a non-negative integer depending on the structure of  $\prod K_a \cdot K_b$  in eq. (3.19). For sufficiently small  $n < 4$ , the singularities of the integrand as some or all  $\alpha$ 's go to zero disappear.

The reasons this regularization preserves the Ward-Takahashi identities of the kind which will be discussed are, firstly, that the vector manipulations such as

$$k^\mu(2p+k)_\mu = [(p+k)^2 - m^2] - (p^2 - m^2),$$

or partial fractioning of a product of two propagators, which are necessary to verify these identities "by hand", are valid in any dimensions, and, secondly, that the shifts of integration variables, dangerous when integrals are divergent, are justified for small enough, or complex  $n$ , since the integral in question is convergent.

The divergence in the original integral is manifested in the poles of  $F_{\Gamma}(n)$  at  $n = 4$ . These poles are removed by the  $R$ -operation, so that  $J_{\Gamma}(n)$  as defined by the  $R$ -operation is finite and well-defined as  $n \rightarrow 4$ . Actually, to our knowledge the proof of this has not appeared in the literature, except for the original discussion of 't Hooft and Veltman. Hepp's proof, for example, does not really apply here, since the analytical discussion of Hepp is not tailored for this kind of regularization. However, the argument of 't Hooft and Veltman is sufficiently convincing and we have no reason to believe why a suitable modification of Hepp's proof, for example, of the BPHZ theorem should not go through with the dimensional regularization.

The above discussion is fine for theories with bosons only. When there are fermions in the theory, a complication may arise. This has to do with the occurrence of the so-called Adler-Bell-Jackiw anomalies. The subject of anomalies in Ward-Takahashi identities has been discussed thoroughly in two excellent lectures by Adler, and by Jackiw, and we shall not go into any further details here. In short, the Adler-Bell-Jackiw anomalies may occur when the verification of certain Ward-Takahashi identities depends on the algebra of Dirac gamma matrices with  $\gamma_5$ , such as  $\gamma_\mu \gamma_5 + \gamma_5 \gamma_\mu = 0$ . Typically, this happens when a proper vertex involving an odd number of axial vector currents cannot

be regularized in a way that preserves all the Ward-Takahashi identities on such a vertex, and as a consequence some of the Ward-Takahashi identities have to be broken. The occurrence of these anomalies is not a matter of not being clever enough to devise a proper regularization scheme: for certain models such a scheme is impossible to devise. The dimensional regularization does not help in such a case, due to the fact that  $\gamma_5$  and the completely antisymmetric tensor density  $\epsilon_{\lambda\mu\nu\rho}$  are unique to four dimensions and do not allow a logically consistent generalization to  $n$  dimensions. When there are anomalies in a spontaneously broken gauge theory, the unitarity of the  $S$ -matrix is in jeopardy since, as we shall see, the unitarity of the  $S$ -matrix, i.e., cancellation of spurious singularities introduced by a particular choice of gauge is inferred from the Ward-Takahashi identities. Gross and Jackiw have shown that, in an Abelian gauge theory, the occurrence of anomalies runs afoul of the dual requirements of unitarity and renormalizability of the theory.

Thus, a satisfactory theory should be free of anomalies. Fortunately, it is possible to construct models which are anomaly-free, by a judicious choice of fermion fields to be included in the model. There are two "lemmas" which make the above assertion possible. One is that the anomalies are not "renormalized", which in particular means that the absence of anomalies in lowest order insures their absence to all orders. This was shown by Adler and Bardeen in the context of an  $SU(3)$  version of the  $\sigma$ -model, and by Bardeen in a more general context which encompasses non-Abelian gauge theories. The second is the observation that all anomalies are related; in particular, if the simplest anomaly involving the vertex of three currents is absent in a model, so are all other anomalies. This can be inferred from an explicit construction of all anomalies by Bardeen, or from a more general and elegant argument of Wess and Zumino.

Let us conclude with a simple example of dimensional regularization: the vacuum polarization in scalar electrodynamics. The Lagrangian is

$$\mathcal{L} = (\partial^\mu \phi^\dagger - ieA^\mu \phi^\dagger)(\partial_\mu \phi + ieA_\mu \phi) - \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - V(\Phi),$$

and the relevant vertices are shown in fig. 3.2. There are two diagrams which contribute to the vacuum polarization, shown in fig. 3.3. The sum of these contributions is

$$I = e^2 \int \frac{d^n k}{(2\pi)^n} \frac{[(2k+p)_\mu (2k+p)_\nu - 2((k+p)^2 - \mu^2)g_{\mu\nu}]}{[(k+p)^2 - \mu^2][k^2 - \mu^2]}. \quad (3.20)$$

We use the exponential parametrization of the propagators to obtain

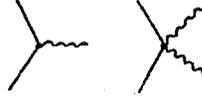


Fig. 3.2. Photon-scalar meson vertices in charged scalar electrodynamics.

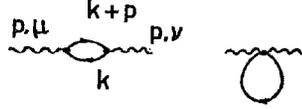


Fig. 3.3. Second order vacuum polarization diagrams in charged scalar electrodynamics.

$$I = \frac{e^2}{i^2} \int_0^\infty d\alpha \int_0^\infty d\beta \int \frac{d^n k}{(2\pi)^n} \exp[i[\alpha(k+p)^2 + \beta k^2 - (\alpha + \beta)(\mu^2 - i\epsilon)]]$$

$$\times [(2k+p)_\mu (2k+p)_\nu - 2((k+p)^2 - \mu^2)g_{\mu\nu}]. \quad (3.21)$$

The exponent is proportional to

$$(\alpha + \beta)k^2 + 2k \cdot p\alpha + \alpha p^2 - (\alpha + \beta)(\mu^2 - i\epsilon)$$

$$= (\alpha + \beta) \left( k + \frac{\alpha}{\alpha + \beta} p \right)^2 + \frac{\alpha\beta}{\alpha + \beta} p^2 - (\alpha + \beta)(\mu^2 - i\epsilon),$$

so we may write

$$I = -e^2 \int_0^\infty d\alpha \int_0^\infty d\beta \int \frac{d^n k}{(2\pi)^n} \exp i \left[ (\alpha + \beta)k^2 + \frac{\alpha\beta}{\alpha + \beta} p^2 - (\alpha + \beta)(\mu^2 - i\epsilon) \right]$$

$$\times \left[ 4k_\mu k_\nu + \left( \frac{\alpha - \beta}{\alpha + \beta} \right)^2 p_\mu p_\nu - g_{\mu\nu} \left\{ 2(k^2 - \mu^2) + \frac{\alpha^2 + \beta^2}{(\alpha + \beta)^2} p^2 \right\} \right]$$

$$= -e^2 (p_\mu p_\nu - p^2 g_{\mu\nu}) \int_0^\infty d\alpha \int_0^\infty d\beta \left( \frac{\alpha - \beta}{\alpha + \beta} \right)^2 \int \frac{d^n k}{(2\pi)^n}$$

$$\times \exp i \left[ (\alpha + \beta)k^2 + \frac{\alpha\beta}{\alpha + \beta} p^2 - (\alpha + \beta)(\mu^2 - i\epsilon) \right]$$

$$\begin{aligned}
 & -e^2 \int_0^\infty d\alpha \int_0^\infty d\beta \int \frac{d^n k}{(2\pi)^n} \exp i \left[ (\alpha + \beta) k^2 + \frac{\alpha\beta}{\alpha + \beta} p^2 - (\alpha + \beta)(\mu^2 - i\epsilon) \right] \\
 & \times \left[ 4k_\mu k_\nu - 2g_{\mu\nu} \left( k^2 - \mu^2 + \frac{\alpha\beta}{(\alpha + \beta)^2} p^2 \right) \right]. \quad (3.22)
 \end{aligned}$$

The first term is explicitly gauge invariant and only logarithmically divergent, so that a subtraction will make it convergent. It is the second term that requires a careful handling. We need the formulas

$$\begin{aligned}
 \int \frac{d^n k}{(2\pi)^n} \exp(i\lambda k^2) &= \frac{i \exp(\frac{1}{4}i\pi n)}{(2\sqrt{\pi\lambda})^n}, \\
 \int \frac{d^n k}{(2\pi)^n} k^2 \exp(i\lambda k^2) &= \frac{1}{i\lambda} \left(-\frac{1}{2}n\right) \frac{i \exp(\frac{1}{4}i\pi n)}{(2\sqrt{\pi\lambda})^n}, \\
 \int \frac{d^n k}{(2\pi)^n} k_\mu k_\nu \exp(i\lambda k^2) &= \frac{1}{n} g_{\mu\nu} \int \frac{d^n k}{(2\pi)^n} k^2 \exp(i\lambda k^2) \\
 &= g_{\mu\nu} \frac{1}{i\lambda} \left(-\frac{1}{2}\right) \frac{i \exp(\frac{1}{4}i\pi n)}{(2\sqrt{\pi\lambda})^n}, \quad (3.23)
 \end{aligned}$$

$i \exp(-\frac{1}{4}i\pi n)$

so that the second term,  $I_2$ , is

$$\begin{aligned}
 I_2 &= -e^2 g_{\mu\nu} \frac{\exp(\frac{1}{4}i\pi n)}{(2\sqrt{\pi})^n} \int_0^\infty d\alpha \int_0^\infty d\beta \frac{1}{(\alpha + \beta)^{n/2}} \\
 & \times \exp \left\{ i \left[ \frac{\alpha\beta}{\alpha + \beta} p^2 - (\alpha + \beta)(\mu^2 - i\epsilon) \right] \right\} \frac{2}{(\alpha + \beta)} \\
 & \times \left\{ i \left( 1 - \frac{1}{2}n \right) - \left[ \frac{\alpha\beta}{\alpha + \beta} p^2 - (\alpha + \beta)\mu^2 \right] \right\} \\
 &= -2ie^2 g_{\mu\nu} \frac{e^{i\pi n/4}}{(2\sqrt{\pi})^n} \int d\alpha d\beta \delta(1 - \alpha - \beta) \int_0^\infty \frac{d\lambda}{\lambda^{n/2-1}} e^{i\lambda[\alpha\beta p^2 - \mu^2 + i\epsilon]} \\
 & \times [\lambda^{-1} (1 - \frac{1}{2}n) + i(\alpha\beta p^2 - \mu^2)]. \quad (3.24)
 \end{aligned}$$

For sufficiently small  $n$ ,  $n < 2$ , the  $\lambda$ -integration is convergent, and

$$\begin{aligned} & \int_0^{\infty} \frac{d\lambda}{\lambda^{n/2-1}} e^{i\lambda(A+i\epsilon)} \left( \frac{1-\frac{1}{2}n}{\lambda} + iA \right) \\ &= \int_0^{\infty} d\lambda \frac{d}{d\lambda} \{ \lambda^{1-n/2} \exp[i\lambda(A+i\epsilon)] \} = 0. \end{aligned} \quad (3.25)$$

So the dimensional regularization gives the gauge invariant result,

$$I_2 = 0.$$

### 3.4. Problem

Repeat the vacuum polarization calculation in spinor electrodynamics using the dimensional regularization.

## 4. The Ward-Takahashi identities

### 4.1. Notations

One of the problems in discussing gauge theories is that notations will get cumbersome if we are to put explicitly space-time variables, Lorentz indices and group representation indices. We will therefore use a highly compact notation. For simplicity in notation, we will assume that the gauge group in question is a simple Lie group. Extension to a product of simple Lie groups, such as  $SU(2) \times U(1)$ , is not too difficult.

We will agree to denote all fields by  $\phi_i$ . Again for simplicity in notation we will assume  $\phi_i$  to be bosons. Inclusion of fermions does not present any difficulty in our discussions, but we will have to be mindful of their anticommuting nature. Thus, for the gauge field  $A_\mu^a(x)$   $i$  stands for the group index  $a$ , the Lorentz index  $\mu$ , and the space-time variable  $x$ . Summation and *integration* over repeated indices will be understood. Thus

$$\phi_i^2 = \int d^4x \sum_a A_\mu^a(x) A^{\mu a}(x) + \dots, \quad (4.1)$$

where the dotted portion includes contribution from other species of fields.

The infinitesimal local gauge transformation may be written as

$$\phi_i \rightarrow \phi'_i = \phi_i + (\Lambda_i^a + t_{ij}^a \phi_j) \theta_a, \quad (4.2)$$

where  $\theta_a = \theta_a(x_a)$  is the space-time dependent parameter of the group  $G$ . We choose  $\phi_i$  to be real, so that

$$t_{ij}^a \equiv \tilde{t}_{ij}^a \delta^4(x_a - x_i) \delta^4(x_a - x_j) \quad (4.3)$$

is real antisymmetric, where  $\tilde{t}_{ij}^a$  is the representation of the generator  $L_a$  of  $G$  in the basis  $\phi_i$ . The inhomogeneous term  $\Lambda_i^a$  is non-vanishing only for the gauge fields

$$\begin{aligned} \Lambda_i^a &= \frac{1}{g} \partial_\mu \delta^4(x_i - x_a) \delta_{a,b} \quad \text{for } \phi_i = A_\mu^a(x_i) \\ &= 0 \quad \text{otherwise.} \end{aligned} \quad (4.4)$$

We shall also define

$$D_i^a = g \Lambda_i^a.$$

Notice that

$$t_{ij}^a D_j^b - t_{ij}^b D_j^a = f^{abc} D_i^c, \quad (4.5)$$

where

$$\begin{aligned} f^{abc} &= \tilde{f}^{abc} \delta^4(x_a - x_b) \delta^4(x_a - x_c), \\ \tilde{t}_{ij}^a \tilde{t}_{jk}^b - \tilde{t}_{ij}^b \tilde{t}_{jk}^a &= \tilde{f}^{abc} t_{ij}^c, \end{aligned} \quad (4.6)$$

$\tilde{f}^{abc}$  being the structure constant of  $G$ . The proof of eq. (4.5) is simple: we will just show

$$t_{ij}^a \Lambda_j^b - t_{ij}^b \Lambda_j^a = f^{abc} \Lambda_i^c.$$

Since  $\Lambda_j^b$  is non-vanishing only when  $j$  refers to a gauge field, let us write  $i = (c, \mu, x_i)$ ,  $j = (d, \nu, x_j)$ . Then

$$t_{ij}^a = g_{\mu\nu} \tilde{f}^{cad} \delta^4(x_i - x_a) \delta^4(x_a - x_j),$$

$$\begin{aligned}
g(t_{ij}^a \Lambda_j^b - t_{ij}^b \Lambda_j^a) &= \tilde{f}^{abc} \int d^4x \left[ \delta^4(x_a - x) \delta^4(x - x_i) \frac{\partial}{\partial x^\mu} \delta^4(x - x_b) \right. \\
&\quad \left. + \delta^4(x_b - x) \delta^4(x - x_i) \frac{\partial}{\partial x^\mu} \delta^4(x - x_a) \right] \\
&= \tilde{f}^{abc} \int d^4x \delta^4(x - x_i) \frac{\partial}{\partial x^\mu} [\delta^4(x_a - x) \delta^4(x - x_b)] \\
&= \tilde{f}^{abc} \frac{\partial}{\partial x^\mu} \int d^4x \delta^4(x - x_i) \delta^4(x - x_a) \delta^4(x - x_b),
\end{aligned}$$

which is equal to  $g$  times the right-hand side of eq. (4.5).

The gauge invariance of the action can be stated in a compact form,

$$D_i^a \frac{\delta S[\phi]}{\delta \phi_i} = 0. \quad (4.7)$$

The linear gauge condition we discussed in lecture 2 may be written as

$$F_a[\phi] = f_{ai} \phi_i, \quad (4.8)$$

where

$$f_{ai} = \partial_i^a \quad \text{for } \phi_i = A_\mu^b(x_i), \quad (4.9)$$

$$\partial_i^a = \delta^{ab} \partial_\mu \delta^4(x_i - x_a). \quad (4.10)$$

In this notation, the effective action is written as

$$S_{\text{eff}}[\phi, \xi, \eta] = S[\phi] - \frac{1}{2\alpha} F_a^2[\phi] + \xi_a M_{ab}[\phi] \eta_b, \quad (4.11)$$

with

$$M_{ab}[\phi] = f_{ai} D_i^b[\phi]. \quad (4.12)$$

#### 4.2. Becchi-Rouet-Stora transformation

The WT identities for gauge theories have been derived in a number of different ways. The most convenient way that I know of is to consider the re-

sponse of the effective action, eq. (4.11), to the so-called Becchi, Rouet, Stora (BRS) transformation. It is a global transformation of anticommuting type which leaves the effective action invariant. Here I shall follow a very elegant discussion of Zinn-Justin given at the Bonn Summer Institute in 1974.

The BRS transformation for non-Abelian gauge theory is defined as

$$\delta \phi_i = D_i^a \eta_a \delta \lambda, \quad (4.13a)$$

$$\delta \eta_a = -\frac{1}{2} g f_{abc} \eta_b \eta_c \delta \lambda, \quad (4.13b)$$

$$\delta \xi_a = -\frac{1}{\alpha} F_a[\phi] \delta \lambda, \quad (4.13c)$$

where  $\delta \lambda$  is an anticommuting constant. Note that if we identify  $\theta_a = \eta_a \delta \lambda$ , we see immediately that the action  $S[\phi]$  is invariant under (4.13a). There are two important properties of the BRS transformation which we shall describe in turn.

(i) The transformations on  $\phi_i$  and  $\eta_a$  are nilpotent, i.e.,

$$\delta^2 \phi_i = 0, \quad (4.14)$$

$$\delta^2 \eta_a = 0. \quad (4.15)$$

*Proof:* Eq. (4.14) follows from

$$\delta(D_i^a \eta_a) = 0. \quad (4.16)$$

Indeed

$$\delta(D_i^a \eta_a) = \frac{\delta D_i^a}{\delta \phi_j} D_j^b \eta_b \delta \lambda \eta_a - \frac{1}{2} g D_i^a f_{abc} \eta_b \eta_c \delta \lambda.$$

Since  $\eta_a$  and  $\eta_b$  anticommute, the coefficient of  $\eta_a \eta_b$  in the first term on the right-hand side may be antisymmetrized with respect to  $a$  and  $b$ . It vanishes as a consequence of eq. (4.5).

To show eq. (4.15), we note that

$$\delta(f_{abc} \eta_b \eta_c) = g f_{abc} f_{bde} \eta_d \eta_e \delta \lambda \eta_c = 0 \quad (4.17)$$

by the Jacobi identity. QED.

(ii) The BRS transformations leave the effective action  $S_{\text{eff}}$  of eq. (4.11) invariant

*Proof:* As noted above  $S[\phi]$  is invariant under eq. (4.13a). We further note that

$$\delta(M_{ab}[\phi]\eta_b) = 0 \quad (4.18)$$

by eq. (4.16) and the definition of  $M_{ab}$ , eq. (4.12). Thus

$$\begin{aligned} & \delta \left[ \xi_a M_{ab} \eta_b - \frac{1}{2\alpha} F_a^2[\phi] \right] \\ &= \frac{1}{\alpha} F_a M_{ab} \eta_b \delta \lambda - \frac{1}{\alpha} F_a \frac{\delta F_a}{\delta \phi_i} D_i^b \eta_b \delta \lambda = 0 \end{aligned} \quad (4.19)$$

by the definition of  $M_{ab}$ , eq. (4.12).

For later use, we remark finally that the metric  $[d\phi_i d\xi_a d\eta_b]$  is invariant under the BRS transformation of eqs. (4.13). I want you to verify it.

#### 4.3. The Ward-Takahashi identities for the generating functional of Green functions

We will first derive the Ward-Takahashi identity satisfied by  $W_F[J]$  of eq. (2.31),

$$Z_F[J] \equiv e^{iW_F[J]} = N \int [d\phi d\xi d\eta] \exp \{ iS_{\text{eff}}[\phi, \xi, \eta] + iJ_i \phi_i \}. \quad (4.20)$$

We first note that, according to the rule of integration over anticommuting numbers

$$\int dc_i c_j = \delta_{ij}, \quad (4.21)$$

we have

$$\int [d\phi d\xi d\eta] \xi_a \exp \{ iS_{\text{eff}}[\phi] + iJ_i \phi_i \} = 0, \quad (4.22)$$

because  $S_{\text{eff}}$  contains  $\xi$  and  $\eta$  only bilinearly. In eq. (4.22) we make a change of variables according to the BRS transformations (4.13). Since a change of integration variables does not change the value of an integral, we have

$$0 = \int [d\phi d\xi d\eta] \left\{ -\frac{1}{\alpha} F_a[\phi] + iJ_i \xi_a D_i^b[\phi] \eta_b \right\} \exp \{ iS_{\text{eff}} + iJ_i \phi_i \}. \quad (4.23)$$

Eq. (4.23) is the WT identity as first derived by Slavnov and Taylor. We can rewrite it in a differential form involving  $Z_F$ . We define

$$(Z_F)_{ba} \equiv Ni \int [d\phi d\xi d\eta] \xi_a \eta_b \exp \{iS_{\text{eff}} + iJ_i \phi_i\}. \quad (4.24)$$

It is the ghost propagator in the presence of external sources  $J_i$ . It satisfies

$$M_{ab} \left[ \frac{1}{i} \frac{\delta}{\delta J} \right] (Z_F)_{bc} = \delta_{ac} Z_F. \quad (4.25)$$

I will leave the derivation of eq. (4.25) as an exercise. Eq. (4.23) can be written as

$$\frac{1}{\alpha} F_a \left[ \frac{1}{i} \frac{\delta}{\delta J} \right] Z_F[J] - J_i D_i^b \left[ \frac{1}{i} \frac{\delta}{\delta J} \right] (Z_F)_{ba} = 0. \quad (4.26)$$

Eq. (4.26) is in the form written down by Zinn-Justin and Lee. It is the WT identity for the generating functional of Green functions, and as such it is rather cumbersome for the discussion of renormalizability, since, as we have seen in lecture 3, the renormalization procedure is phrased in terms of (single-particle irreducible) proper vertices. Nevertheless, eq. (4.26) was used to deduce "by hand" consequences of gauge symmetry or renormalization parts by Zinn-Justin and myself. We do not have to do this, since we know better now. Eq. (4.26) will be useful in discussing the unitarity of the  $S$ -matrix later, however.

#### 4.4. The Ward-Takahashi identities: inclusion of ghost sources

To discuss renormalization of gauge theories, we have to consider proper vertices some of whose external lines are ghosts. For this reason, the ghost fields  $\xi$  and  $\eta$  should have their own sources. We therefore return to eq. (2.34),

$$\begin{aligned} Z[J_\mu, \beta, \beta^\dagger] &= N \int [d\phi d\xi d\eta] \\ &\times \exp [i \{S_{\text{eff}}[\phi, \xi, \eta] + \xi \beta + \beta^\dagger \eta + J_i \phi_i\}]. \end{aligned} \quad (4.27)$$

For the ensuing discussion, it is more convenient to consider an object

$$\Sigma[\phi, \xi, \eta, K, L] = S_{\text{eff}}[\phi, \xi, \eta] + K_i D_i^a[\phi] \eta_a + \frac{1}{2} g L_a f_{abc} \eta_b \eta_c, \quad (4.28)$$

and define

$$Z[J, \beta, \beta^\dagger, K, L] = N \int [d\phi d\xi d\eta] \times \exp[i\{\Sigma[\phi, \xi, \eta, K, L] + \xi\beta + \beta^\dagger\eta + J_i\phi_i\}]. \quad (4.29)$$

In eq. (4.28),  $K_i$  and  $L_a$  are sources for the composite operators  $D_i^a[\phi]\eta_a$  and  $\frac{1}{2}gf_{abc}\eta_b\eta_c$ , respectively. It follows from eqs. (4.16) and (4.17) that  $\Sigma$  is invariant under the BRS transformations (4.13). Note further that  $K_i$  is of anti-commuting type, and

$$\frac{\delta \Sigma}{\delta K_i} = D_i^a[\phi]\eta_a, \quad (4.30)$$

$$\frac{\delta \Sigma}{\delta L_a} = \frac{1}{2}gf_{abc}\eta_b\eta_c. \quad (4.31)$$

The invariance of  $\Sigma$  is expressed as

$$\left\{ D_i^a \eta_a \delta \lambda \frac{\delta}{\delta \phi_i} - \frac{1}{2}gf_{abc}\eta_b\eta_c \delta \lambda \frac{\delta}{\delta \eta_a} - F_a[\phi] \delta \lambda \frac{\delta}{\delta \xi_a} \right\} \Sigma = 0,$$

or

$$\frac{\delta \Sigma}{\delta \phi_i} \frac{\delta \Sigma}{\delta K_i} + \frac{\delta \Sigma}{\delta L_a} \frac{\delta \Sigma}{\delta \eta_a} + F_a \frac{\delta \Sigma}{\delta \xi_a} = 0. \quad (4.32)$$

We need one more equation,

$$\frac{\delta \Sigma}{\delta \xi} = f_{ai} \frac{\delta \Sigma}{\delta K_i}. \quad (4.33)$$

Let us examine the consequences of eqs. (4.32) and (4.33). We perform a change of variables

$$\delta \phi_i = \frac{\delta \Sigma}{\delta K_i} \delta \lambda, \quad (4.34a)$$

$$\delta \eta_a = -\frac{\delta \Sigma}{\delta L_a} \delta \lambda, \quad (4.34b)$$

$$\delta \xi_a = -\frac{1}{\alpha} F_a \delta \lambda. \quad (4.34c)$$

Eq. (4.32) tells us that  $\Sigma$  is invariant under such a transformation, and the integration measure  $[d\phi_i d\xi d\eta]$  is also, thanks to

$$\frac{\delta^2 \Sigma}{\delta \phi_i \delta K_i} = 0, \quad (4.35a)$$

$$\frac{\delta^2 \Sigma}{\delta \eta_a \delta L_a} = 0. \quad (4.35b)$$

Thus, the change of variables (4.34) in eq. (4.29) leads to

$$\int [d\phi d\xi d\eta] \left\{ \frac{\delta \Sigma}{\delta K_i} J_i - \frac{\delta \Sigma}{\delta L_a} \beta_a^\dagger + \frac{1}{\alpha} F_a \beta_a \right\} \\ \times \exp[i\{\Sigma + \xi \cdot \beta + \beta^\dagger \cdot \eta + J_i \phi_i\}] = 0. \quad (4.36)$$

Next, the equation of motion for  $\eta$  is

$$\int [d\phi d\xi d\eta] \left\{ \frac{\delta \Sigma}{\delta \xi_a} + \beta_a \right\} \exp[i\{\Sigma + \xi \cdot \beta + \beta^\dagger \cdot \eta + J_i \phi_i\}] = 0. \quad (4.37)$$

Combining eqs. (4.33) and (4.37), we obtain

$$\int [d\phi d\xi d\eta] \left[ f_{ai} \frac{\delta \Sigma}{\delta K_i} + \beta_a \right] \exp[i\{\Sigma + \xi \cdot \beta + \beta^\dagger \cdot \eta + J_i \phi_i\}] = 0. \quad (4.38)$$

Eqs. (4.36) and (4.38) are the basis for deriving the WT identity for the generating functional of proper vertices.

#### 4.5. The Ward-Takahashi identities for the generating functional of proper vertices

The generating functional of proper vertices is obtained from  $W$ ,

$$W = -i \ln Z,$$

by a Legendre transformation. We define

$$\phi_i = \frac{\delta W}{\delta J_i}, \quad (4.39a)$$

$$\eta_a = \frac{\delta W}{\delta \beta_a^\dagger}, \quad (4.39b)$$

$$\xi_a = -\frac{\delta W}{\delta \beta_a}, \quad (4.39c)$$

where we have used the same symbols for the expectation values of fields as for the integration variables. The generating functional for proper vertices is

$$\tilde{\Gamma}[\phi, \xi, \eta, K, L] = W[J, \beta, \beta^\dagger, K, L] - J_i \phi_i - \xi \cdot \beta - \beta^\dagger \cdot \eta. \quad (4.40)$$

As usual, we have the relations dual to eqs. (4.39),

$$-J_i = \frac{\delta \tilde{\Gamma}}{\delta \phi_i}, \quad (4.41a)$$

$$\beta_a^\dagger = \frac{\delta \tilde{\Gamma}}{\delta \eta_a}, \quad (4.41b)$$

$$-\beta_a = \frac{\delta \tilde{\Gamma}}{\delta \xi_a}. \quad (4.41c)$$

It is easy to verify that if  $W$  and  $\tilde{\Gamma}$  depend on parameters  $Q$ , such as  $K$  or  $L$  in our case, which are not involved in the Legendre transformation, they satisfy

$$\frac{\delta \tilde{\Gamma}}{\delta Q} = \left. \frac{\delta W}{\delta Q} \right|_{J=J[\phi]}. \quad (4.42)$$

From eqs. (4.36) and (4.38) we can derive two equations satisfied by  $\tilde{\Gamma}$ ,

$$\frac{\delta \tilde{\Gamma}}{\delta K_i} \frac{\delta \tilde{\Gamma}}{\delta \phi_i} + \frac{\delta \tilde{\Gamma}}{\delta L_a} \frac{\delta \tilde{\Gamma}}{\delta \eta_a} + \frac{1}{\alpha} F_a \frac{\delta \tilde{\Gamma}}{\delta \xi_a} = 0 \quad (4.43)$$

$$f_a \frac{\delta \tilde{\Gamma}}{\delta K_i} = \frac{\delta \tilde{\Gamma}}{\delta \xi_a}. \quad (4.44)$$

It is important to observe the correspondence between eqs. (4.32) and (4.33), and eqs. (4.43) and (4.44).

If we now define

$$\Gamma[\phi, \xi, \eta, K, L] = \tilde{\Gamma}[\phi, \xi, \eta, K, L] + \frac{1}{2\alpha} F_a^2[\phi], \quad (4.45)$$

we have

$$\frac{\delta \Gamma}{\delta K_i} \frac{\delta \Gamma}{\delta \phi_i} + \frac{\delta \Gamma}{\delta L_a} \frac{\delta \Gamma}{\delta \eta_a} = 0. \quad (4.46)$$

$$f_{ai} \frac{\delta \Gamma}{\delta K_i} = \frac{\delta \Gamma}{\delta \xi_a}. \quad (4.47)$$

The functional  $\Gamma$  carries a net ghost number zero, where we define the ghost number  $N_g$  as

$$\begin{aligned} N_g[\eta] &= 1, & N_g[K] &= -1, \\ N_g[\xi] &= -1, & N_g[L] &= -2, \\ N_g[\phi] &= 0. \end{aligned}$$

Clearly  $\Gamma$  may be expanded in terms of the ghost number carrying fields:

$$\Gamma[\phi, \xi, \eta, K, L] = \Gamma_0[\phi] + \xi_a \Gamma_{ab}[\phi] \eta_b + K_i \Gamma_{ib}[\phi] \eta_b + \dots \quad (4.48)$$

Substituting the expression (4.48) in eqs. (4.46) and (4.47), differentiating with respect to  $\eta_b$  and setting all ghost number carrying fields equal to zero, we obtain

$$\Gamma_{ib}[\phi] \frac{\delta \Gamma_0[\phi]}{\delta \phi_i} = 0, \quad (4.49)$$

$$f_{ai} \Gamma_{ib}[\phi] = \Gamma_{ab}[\phi]. \quad (4.50)$$

These are the equations first derived by me from (4.26) by a complicated functional manipulation. These are the fully dressed versions of eqs. (4.7) and (4.12),

$$\Gamma_{ib}[\phi] \leftrightarrow D_i^b[\phi],$$

$$\Gamma_0[\phi] \leftrightarrow S[\phi],$$

$$\Gamma_{ab}[\phi] \leftrightarrow M_{ab}[\phi].$$

#### 4.6. Problems

4.6.1. Convince yourself that the measure  $[d\phi d\xi d\eta]$  is invariant under the BRS transformation.

4.6.2. Show that

$$\int (\prod dc_i) c_k \frac{\partial}{\partial c_l} f(c) = \delta_{kl} \int (\prod dc_i) f(c).$$

Prove eq. (4.25).

4.6.3. Show that

$$dc_i \frac{\partial}{\partial c_i} f(c) = 0, \quad i \text{ not summed,}$$

where  $c_i$  is an anticommuting number. Prove eq. (4.37).

4.6.4. Derive the WT identity for  $Z[J, \beta, \beta^\dagger]$ ,

$$Z[J, \beta, \beta^\dagger] = N \int [d\phi d\xi d\eta] \prod_a \delta(F_a[\phi]) \\ \times \exp[i\{S[\phi] + \xi_a M_{ab}[\phi] \eta_b + J_i \phi_i + \beta^\dagger \cdot \eta + \xi \cdot \beta\}].$$

(Zinn-Justin.)

### 5. Renormalization of pure gauge theories

#### 5.1. Renormalization equation

We are ready to discuss renormalization of non-Abelian gauge theories based on the WT identity for proper vertices derived in the last lecture.

Let us recall that our Feynman integrals are regularized dimensionally so that for a suitably chosen  $n$  not equal to 4, all integrals are convergent. Thus, we can perform the Bogoliubov  $R$ -operation after the integral has been done, instead of making the subtraction of eq. (3.10) in the integrand as Zimmermann dictates. In fact this is the procedure used by Bogoliubov, Parasiuk and Hepp. Further, instead of making subtractions at  $p_i = 0$ , we will choose a point where all momenta flowing into a renormalization part are Euclidian.

For a vertex with  $n$  external lines, this point may be chosen to be  $-p_i^2 = a^2$ ,  $p_i \cdot p_j = a^2/(n-1)$ . This is to avoid infrared divergences. At this point the square of a sum of any subset of momenta is always negative, so that the amplitude is real and free of singularities.

For simplicity we first consider a pure gauge theory. Inclusion of matter fields, such as scalar and spinor with renormalizable interactions present no difficulty. In particular, coupling of gauge fields with scalar mesons will be treated in chapter 6.

We may write down the proper vertex as a sum of terms, each being a product of a scalar function of external momenta and a tensor covariant, which is a polynomial in the components of external momenta carrying available Lorentz indices. All renormalization parts in this theory have either  $D = 0$  or 1. The self-mass of a gauge boson is purely transverse as we shall see, so that it also has effectively  $D = 0$ . Thus, only the scalar functions associated with tensor covariants of lowest order are divergent as  $n \rightarrow 4$ . (Note also that vertices involving external ghost lines have lower superficial degrees of divergence than simple power counting indicates. This is because  $\xi_a$  always appears as  $\partial^\mu \xi_a$ .)

The basic proposition on renormalization of a gauge theory is the following. If we scale fields and the coupling constant according to

$$\begin{aligned} \phi_i &= Z^{1/2}(\epsilon)\phi_i^r, & K_i &= \tilde{Z}^{1/2}(\epsilon)K_i^r, \\ \xi_a &= \tilde{Z}^{1/2}(\epsilon)\xi_a^r, & L_a &= Z^{1/2}(\epsilon)L_a^r, \\ \eta_a &= \tilde{Z}^{1/2}(\epsilon)\eta_a^r, & \alpha &= Z(\epsilon)\alpha^r, \\ g &= \frac{X(\epsilon)}{\tilde{Z}(\epsilon)Z^{1/2}(\epsilon)}g^r, \end{aligned} \tag{5.1}$$

where  $\epsilon = n - 4$  is the regularization parameter, and choose  $Z(\epsilon)$ ,  $X(\epsilon)$  and  $\tilde{Z}(\epsilon)$  appropriately, then

$$\tilde{\Gamma}^r[\phi^r, \xi^r, \eta^r, K^r, L^r, \alpha^r, g^r] = \Gamma[\phi, \xi, \eta, K, L, \alpha; g] \tag{5.2}$$

is a finite (that is, as  $\epsilon = D - 4 \rightarrow 0$ ) functional of its arguments  $\phi^r$ ,  $\xi^r$ ,  $\eta^r$ ,  $K^r$ ,  $L^r$  and  $\alpha^r$ . Under the renormalization transformation of (5.1), eqs. (4.45) and (4.46) become

$$\frac{\delta \Gamma^r}{\delta K_i^r} \frac{\delta \Gamma^r}{\delta \phi_i^r} + \frac{\delta \Gamma^r}{\delta L_a^r} \frac{\delta \Gamma^r}{\delta \eta_a^r} = 0, \tag{5.3}$$

$$F_{ai} \frac{\delta \Gamma^r}{\delta K_i^r} = \frac{\delta \Gamma^r}{\delta \xi_a^r}. \quad (5.4)$$

We will expand loopwise,

$$\tilde{\Gamma} = \sum_{n=0}^{\infty} \tilde{\Gamma}_{(n)}. \quad (5.5)$$

We have

$$\tilde{\Gamma}_{(0)} = \Sigma = \Sigma_0 - \frac{1}{2\alpha} F_a^2[\phi]. \quad (5.6)$$

Suppose that our basic proposition is true up to the  $(n-1)$  loop approximation. That is, up to this order, all divergences are removed by rescaling of fields and parameters as in eq. (5.1). We suppose that we have determined the renormalization constants up to this order,

$$\begin{aligned} (Z)_{n-1} &= 1 + z_{(1)} + \dots + z_{(n-1)}, \\ (\tilde{Z})_{n-1} &= 1 + \tilde{z}_{(1)} + \dots + \tilde{z}_{(n-1)}, \\ (X)_{n-1} &= 1 + x_{(1)} + \dots + x_{(n-1)}. \end{aligned} \quad (5.7)$$

We have to show that the divergences in the  $n$ -loop approximation are also removed by suitably chosen  $z_n$ ,  $\tilde{z}_n$  and  $x_n$ .

Following Zinn-Justin, we introduce the symbol

$$\Gamma^r * \Gamma^r \equiv \frac{\delta \Gamma^r}{\delta K_i^r} \frac{\delta \Gamma^r}{\delta \phi_i^r} + \frac{\delta \Gamma^r}{\delta L_a^r} \frac{\delta \Gamma^r}{\delta \eta_a^r} = 0, \quad (5.8)$$

where the superscript  $r$  denotes here the quantities renormalized up to the  $(n-1)$  loop approximation. We can write eq. (5.8) as

$$\Gamma_{(n)}^r * \Gamma_{(0)}^r + \Gamma_{(0)}^r * \Gamma_{(n)}^r = -\Gamma_{(n-1)}^r * \Gamma_{(1)}^r - \Gamma_{(1)}^r * \Gamma_{(n-1)}^r - \dots, \quad (5.9)$$

with

$$\Gamma_{(0)}^r = \Sigma[\phi^r, \xi^r, \eta^r, K^r, L^r, g^r, \alpha^r]. \quad (5.10)$$

The right-hand side of eq. (5.9) involves only quantities with less than  $n$  loops, it is finite by the induction hypothesis. Further, divergences in subdiagrams of  $\Gamma_{(n)}$  are removed by renormalizations up to  $(n - 1)$  loops. Thus, the only remaining divergences in  $\Gamma_{(n)}$  are the overall ones. Let us denote by  $\{\}^{\text{div}}$  the divergent part. If we adjust finite parts of  $\Gamma_{(n)}^r$  appropriately, we have

$$\{\Gamma_{(n)}^r\}^{\text{div}} * \Gamma_{(0)}^r + \Gamma_{(0)}^r * \{\Gamma_{(n)}^r\}^{\text{div}} = 0, \quad (5.11)$$

$$\partial_i^a \frac{\delta}{\delta K_i^r} \{\Gamma_{(n)}^r\}^{\text{div}} = \frac{\delta}{\delta \xi_a^r} \{\Gamma_{(n)}^r\}^{\text{div}}. \quad (5.12)$$

### 5.2. Solution to renormalization equation

The divergent part of  $\Gamma_{(n)}^r$  is a solution of the functional differential equations (5.11), (5.12). We recall that

$$\begin{aligned} \Gamma_{(0)}^r &= \Sigma_0[\phi^r, \xi^r, \eta^r, K^r, g^r] \\ &= S[\phi^r, g^r] + \xi_a^r \partial_i^a D_i^b [\phi^r, g^r] \eta_b^r + K_i^r D_i^r \eta_b^r \\ &\quad + \frac{1}{2} g^r L_a^r f_{abc} \eta_b^r \eta_c^r. \end{aligned} \quad (5.13)$$

We can write eqs. (5.11), (5.12) as

$$\mathcal{Q} \{\Gamma_{(n)}^r\} = 0, \quad (5.14)$$

$$\left( \partial_i^a \frac{\delta}{\delta K_i^r} - \frac{\delta}{\delta \xi_a^r} \right) \{\Gamma_{(n)}^r\}^{\text{div}} = 0, \quad (5.15)$$

where the functional operator  $\mathcal{Q}$  is given by

$$\mathcal{Q} = \mathcal{Q}_0 + \mathcal{Q}_1, \quad (5.16)$$

$$\mathcal{Q}_0 = \frac{\delta \Gamma_{(0)}^r}{\delta K_i^r} \frac{\delta}{\delta \phi_i^r} + \frac{\delta \Gamma_{(0)}^r}{\delta L_a^r} \frac{\delta}{\delta \eta_a^r}, \quad (5.17a)$$

$$\mathcal{Q}_1 = \frac{\delta \Gamma_{(0)}^r}{\delta \phi_i^r} \frac{\delta}{\delta K_i^r} + \frac{\delta \Gamma_{(0)}^r}{\delta \eta_a^r} \frac{\delta}{\delta L_a^r}. \quad (5.17b)$$

From now on, for the interest of notational economy, we will drop the superscript  $r$ , until further notice.

An important aid in solving eq. (5.14) is the observation that

$$\mathcal{G}^2 = 0. \quad (5.18)$$

We will prove this in steps. First, we verify by direct computation that

$$\mathcal{G}_0^2 = 0, \quad (5.19)$$

$$\mathcal{G}_0 = D_i^a \eta_a \frac{\delta}{\delta \phi_i} + \frac{1}{2} f_{abc} \eta_b \eta_c \frac{\delta}{\delta \eta_a}.$$

Eq. (5.19) is a direct consequence of eqs. (5.16), (5.17) that the BRS transformation on  $\phi_i$  and  $\eta_a$  is nilpotent. Next we note that

$$\begin{aligned} \{\mathcal{G}_0, \mathcal{G}_1\} &= \left[ \mathcal{G}_0, \frac{\delta \Gamma_{(0)}}{\delta \phi_i} \right] \frac{\delta}{\delta K_i} + \left( \mathcal{G}_0, \frac{\delta \Gamma_{(0)}}{\delta \eta_a} \right) \frac{\delta}{\delta L_a} \\ &= - \left( \frac{\delta \mathcal{G}_0}{\delta \phi_i} \Gamma_{(0)} \right) \frac{\delta}{\delta K_i} + \left( \frac{\delta \mathcal{G}_0}{\delta \eta_a} \Gamma_{(0)} \right) \frac{\delta}{\delta L_a}, \end{aligned} \quad (5.20)$$

where we have used the fact that

$$\mathcal{G}_0 \Gamma_{(0)} = 0. \quad (5.21)$$

We note also

$$\begin{aligned} \mathcal{G}_1^2 &= \frac{\delta \Gamma_{(0)}}{\delta \phi_j} \frac{\delta^2 \Gamma_{(0)}}{\delta K_j \delta \phi_i} \frac{\delta}{\delta K_i} \\ &+ \left( \frac{\delta \Gamma_{(0)}}{\delta \phi_j} \frac{\delta^2 \Gamma_{(0)}}{\delta K_j \delta \eta_a} + \frac{\delta \Gamma_{(0)}}{\delta \eta_b} \frac{\delta^2 \Gamma_{(0)}}{\delta L_b \delta \eta_a} \right) \frac{\delta}{\delta L_a}. \end{aligned} \quad (5.22)$$

Direct computations yield

$$\frac{\delta \mathcal{G}_0}{\delta \phi_i} \Gamma_{(0)} = \frac{\delta \Gamma_{(0)}}{\delta \phi_j} \frac{\delta^2 \Gamma_{(0)}}{\delta K_j \delta \phi_i} = \eta_a \frac{\delta D_j^a}{\delta \phi_i} \frac{\delta \Gamma_{(0)}}{\delta \phi_j},$$

$$\begin{aligned} \frac{\delta \mathcal{G}_0}{\delta \eta_a} \Gamma^{(0)} &= - \left( \frac{\delta \Gamma^{(0)}}{\delta \phi_j} \frac{\delta^2 \Gamma^{(0)}}{\delta K_j \delta \eta_a} + \frac{\delta \Gamma^{(0)}}{\delta \eta_b} \frac{\delta^2 \Gamma^{(0)}}{\delta L_b \delta \eta_a} \right) \\ &= D_j^a \frac{\delta \Gamma^{(0)}}{\delta \phi_j} + f_{bac} \eta_c \frac{\delta}{\delta \eta_b}. \end{aligned}$$

Thus

$$\{\mathcal{G}_0, \mathcal{G}_1\} + \mathcal{G}_1^2 = 0,$$

which proves eq. (5.18).

The fact that  $\mathcal{G}$  is nilpotent means that, in general,  $\mathcal{G}\mathcal{F}$  for arbitrary  $\mathcal{F} = \mathcal{F}(\phi, \xi, \eta, K, L)$  is a solution of eq. (5.14)

$$\mathcal{G}(\mathcal{G}\mathcal{F}) = 0. \tag{5.23}$$

The question is whether there are other solutions not of the form  $\mathcal{G}\mathcal{F}$ . This question also arises in renormalization of gauge invariant operators, and has been studied, in particular, by Kluberg-Stern and Zuber. They also advanced a conjecture: they suggested that the general solution to eq. (5.14) is of the form

$$\{\Gamma_{(n)}^r\}^{\text{div}} = G[\phi] + \mathcal{G}\mathcal{F}[\phi, \xi, \eta, K, L], \tag{5.24}$$

where  $G[\phi]$  is a gauge invariant functional,

$$D_i^a[\phi] \frac{\delta}{\delta \phi_i} G[\phi] = 0. \tag{5.25}$$

Recently, Joglekar and I were able to prove this, mostly by the effort of the first author. The proof is tedious, and I believe that it can be improved as to rigor, elegance and length. For this reason, I will not present the proof. It is easy to see that the form (5.24) satisfies eq. (5.14) and that  $G[\phi]$  of eq. (5.25) is not expressible as  $\mathcal{G}\mathcal{F}$ , in this case at least. It is the completeness of eq. (5.24) which requires proof.

Eq. (5.15) [or (5.4)] is immediately solved. It means that

$$\Gamma_{(n)}^r[\phi, \xi, \eta, K, L] = \Gamma_{(n)}^r[\phi, 0, \eta, K_i + \partial_i^\alpha \xi_a, L] + K_i Q^i[\phi, \xi, \eta, L], \tag{5.26}$$

where  $Q^i$  is transverse:  $\partial_i^\alpha Q_i = 0$ .

The quantity  $\{\Gamma_{(n)}^r\}^{\text{div}}$  is a local functional of its arguments. If we assign to  $K$  and  $L$  the dimensions  $D(K) = 2$  and  $D(L) = 2$ , then  $\Sigma_0$  has the uniform dimension 0, and so does  $\{\Gamma_{(n)}^r\}^{\text{div}}$ . It has  $N_g[\Gamma_{(n)}^r] = 0$ . Since  $N_g[\mathcal{G}] = +1$ , it follows that  $N_g[\mathcal{F}] = -1$  in eq. (5.24). In order that the right-hand side of eq. (5.24) is local, both  $\mathcal{G}$  and  $\mathcal{F}$  must be separately local. The most general form of  $\{\Gamma_{(n)}^r\}^{\text{div}}$  satisfying the above requirements is

$$\{\Gamma_{(n)}^r\}^{\text{div}} = \alpha(\epsilon)S[\phi] + \mathcal{G} \{ \beta(\epsilon)(K_i + \partial_i^a \xi_a)\phi_i + \gamma(\epsilon)L_a \eta_a \}, \quad (5.27)$$

where  $\alpha, \beta, \gamma$  are in general divergent, i.e.,  $\epsilon$ -dependent, constants. Using the explicit form of  $\mathcal{G}$ , eqs. (5.17), we can write

$$\begin{aligned} \{\Gamma_{(n)}^r\}^{\text{div}} &= \alpha S[\phi] - \beta(K_i + \partial_i^a \xi_a)D_i^b \eta_b \\ &+ \gamma L_a \frac{\delta \Gamma_{(0)}}{\delta L_a} + \beta \frac{\delta \Gamma_{(0)}}{\delta \phi_i} \phi_i + \gamma \frac{\delta \Gamma_{(0)}}{\delta \eta_a} \eta_a. \end{aligned} \quad (5.28)$$

In eq. (5.27)  $G[\phi]$  is equal to  $S[\phi]$ . This is so because the action is the only local functional of dimension four which satisfies eq. (5.25).

Because

$$S[\phi, g] = \frac{1}{g^2} S[g\phi, 1],$$

we have

$$S[\phi, g] = \frac{1}{2} \phi_i \frac{\delta S}{\delta \phi_i} - \frac{1}{2} g \frac{\partial S}{\partial g}. \quad (5.29)$$

Thus, combining eqs. (5.28), (5.29), we obtain

$$\begin{aligned} \{\Gamma_{(n)}^r\}^{\text{div}} &= \left\{ \left( \frac{1}{2} \alpha + \beta \right) \left( \phi_i \frac{\delta}{\delta \phi_i} + L_a \frac{\delta}{\delta L_a} \right) \right. \\ &\quad \left. - \frac{1}{2} (\gamma + \beta) \left[ \eta_a \frac{\delta}{\delta \eta_a} + \xi_a \frac{\delta}{\delta \xi_a} + K_i \frac{\delta}{\delta K_i} \right] \right. \\ &\quad \left. - \frac{1}{2} \alpha g \frac{\partial}{\partial g} \right\} \Gamma_{(0)}[\phi, \xi, \eta, K, L, g]. \end{aligned} \quad (5.30)$$

Recall that in eq. (5.30),  $\phi, \xi, \eta, K, L$  and  $g$  are renormalized quantities up to the  $(n-1)$  loop approximation. We shall denote them by  $(\phi^r)_{n-1}$ , etc.

If we now define  $(Z)_n, (\tilde{Z})_n, (X)_n$  by

$$(Z)_n = (Z)_{n-1} + z_{(n)}, \quad \text{etc.}, \quad (5.31)$$

and renormalize the fields and coupling constant according to

$$(\phi_i^r)_n = \frac{Z_n^{1/2}}{Z_{n-1}^{1/2}} (\phi_i^r)_{n-1}, \quad \text{etc.}, \quad (5.32)$$

$$(g^r)_n = \frac{(X)_n}{(\tilde{Z})_n (Z)_n^{1/2}} \left[ \frac{(X)_{n-1}}{(\tilde{Z})_{n-1} (Z)_{n-1}^{1/2}} \right]^{-1} (g^r)_{n-1} \quad (5.33)$$

and choose  $z_{(n)}$ ,  $\tilde{z}_{(n)}$ ,  $x_{(n)}$  to be

$$\begin{aligned} z_{(n)} &= -\left(\frac{1}{2}\alpha + \beta\right)(\epsilon), \\ \tilde{z}_{(n)} &= \frac{1}{2}(\gamma + \beta)(a), \\ x_{(n)} - \tilde{z}_{(n)} - \frac{1}{2}z_{(n)} &= \frac{1}{2}\alpha(\epsilon), \end{aligned} \quad (5.34)$$

then  $\{\Gamma_{(n)}^r\}^{\text{div}}$  is eliminated:  $\Gamma_{(n)}^r$  is a finite functional in terms of  $(\phi^r)_n, \dots$ , and  $(g^r)_n$ . Furthermore, since  $(\phi^r)_n = (Z)_n^{1/2}\phi, \dots$ , eqs. (5.3), (5.4) are also true for the newly renormalized quantities. This completes the induction.

Note further that

$$\tilde{\Gamma}^r[\phi^r, \xi^r, \eta^r, K^r, L^r, g] = \Gamma^r - \frac{1}{2\alpha^r} (\partial_i^a \phi_i)^2 \quad (5.35)$$

is finite. The renormalized field  $\phi^r$  transforms under the gauge transformation as

$$\phi_i^r \rightarrow \phi_i^r + \left[ \partial_i^a + g^r \left( \frac{X}{\tilde{Z}} \right) t_{ij}^a \phi_j^r \right] \theta_a.$$

## 6. Renormalization of theories with spontaneously broken symmetry

### 6.1. Inclusion of scalar fields

In chapter 5, we detailed the renormalization of pure gauge theories. Let us consider now a theory of gauge bosons and scalar mesons. Let

$$\phi_i = (A_\mu^a(x), s_\alpha(x)),$$

where  $s_\alpha$  are scalar mesons, and let

$$f_{bi} = (\delta_{ab} \partial_\mu, c_{b\alpha}) \delta^4(x_b - x_i)$$

be the vector which defines the gauge. The action for the scalar fields is of the form

$$S_{\text{matter}}[s, A] = \frac{1}{2}(Ds)_\alpha^2 - V(s), \quad (6.1)$$

where  $(Ds)_\alpha$  is the covariant derivative acting on  $s$ ,  $V(s)$  is a  $G$ -invariant quartic polynomial in  $s$  and of dimension at most zero.

We shall write

$$\left. \frac{\delta^2 V(s)}{\delta s_\alpha \delta s_\beta} \right|_{s=0} \equiv M_{\alpha\beta}^r + (\delta M)_{\alpha\beta}, \quad (6.2)$$

$$[M^r, t^a] = [\delta M, t^a] = 0, \quad (6.3)$$

where  $M_{\alpha\beta}^r$  is the renormalized mass matrix for the scalar mesons. We shall assume for the moment that  $M^r$  is a positive semi-definite matrix.

Let us discuss renormalization. Almost everything we discussed in the last section holds true. In particular we have

$$\{\Gamma_{(n)}^r\}^{\text{div}} = G[\phi] + \mathcal{G}\mathcal{F}[\phi, \xi, \eta, K, L],$$

$$D_t^a[\phi] \frac{\delta}{\delta \phi_t} G[\phi] = 0,$$

$$\left\{ \partial_t^a \frac{\delta}{\delta K_t} + C_\alpha^a \frac{\delta}{\delta K_\alpha} - \frac{\delta}{\delta \xi^a} \right\} \{\Gamma_{(n)}^r\}^{\text{div}} = 0,$$

where we have written  $\phi_t = \{A_t, s_\alpha\}$  and  $K_t = \{K_t, K_\alpha\}$ ,  $A_t$  being the gauge fields,  $A_t = A_\mu^a(x)$ ,  $t = (a, \mu, x)$ . Now we have

$$G[\phi] = \alpha_1 S_{\text{gauge}}[A] + \alpha_2 \frac{1}{2}(Ds)_\alpha^2 - V'(s), \quad (6.4)$$

where  $V'$  is a  $G$ -invariant quartic polynomial in  $s$ . This term is eliminated by renormalizations of coupling constants appearing in  $V(s)$ .  $\mathcal{F}$  takes the form

$$\mathcal{F} = \beta_1(K_t + \partial_t^a \xi_a)A_t + \beta_2(K_\alpha + c_\alpha^a \xi_a)s_\alpha + \gamma L_a \eta_a + \delta(K_\alpha + c_\alpha^a \xi_a)d_{\alpha\beta}^b c_\beta^b,$$

where  $d_{\alpha\beta}^b$  is a  $G$ -covariant coefficient. (It could be  $t_{\alpha\beta}^b$ , or something else, such as the  $d$ -type coupling in  $SU(3)$ , for example.) This gives

$$\begin{aligned} \{\Gamma_{(n)}^r\}^{\text{div}} = & \left( \frac{1}{2}\alpha_1 + \beta_1 \right) \left[ A_t \frac{\delta}{\delta A_t} + L_a \frac{\delta}{\delta L_a} \right] - \frac{1}{2}\alpha_1 g \frac{\partial}{\partial g} \quad (6.5) \\ & + \left( \frac{1}{2}\alpha_2 + \beta_2 \right) s_\alpha \frac{\delta}{\delta s_\alpha} + v_\alpha(\epsilon) \frac{\delta}{\delta s_\alpha} + \left( \frac{1}{2}\alpha_1 + \beta_1 - \left( \frac{1}{2}\alpha_2 + \beta_2 \right) - \left( \frac{1}{2}\beta_1 + \gamma \right) \right) K_\alpha \frac{\delta}{\delta K_\alpha} \\ & - \frac{1}{2}(\beta_1 + \gamma) \left[ K_t \frac{\delta}{\delta K_t} + \eta_a \frac{\delta}{\delta \eta_a} + \xi_a \frac{\delta}{\delta \xi_a} \right] + \left[ \left( \frac{1}{2}\alpha_1 + \beta_1 \right) - \left( \frac{1}{2}\alpha_2 + \beta_2 \right) \right] c_\alpha^a \frac{\partial}{\partial c_\alpha^a} \Gamma_{(0)}^r, \end{aligned}$$

where

$$v_\alpha(\epsilon) = \delta(\epsilon) d_{\alpha\beta}^a c_\beta^a.$$

These divergences are eliminated if we renormalize  $A$ ,  $\xi$ ,  $\eta$ ,  $K_t$ ,  $L_a$  and  $g$  as before and

$$s_\alpha = Z_s^{1/2} s_\alpha^r, \quad K_\alpha = \left( \frac{\tilde{Z}Z}{Z_s} \right)^{1/2} K_\alpha^r, \quad c_\alpha^a = \left( \frac{Z}{Z_s} \right)^{1/2} (c_\alpha^a)^r,$$

which leaves

$$\begin{aligned} \frac{\delta \Gamma}{\delta K_t} \frac{\delta \Gamma}{\delta A_t} + \frac{\delta \Gamma}{\delta K_\alpha} \frac{\delta \Gamma}{\delta s_\alpha} + \frac{\delta \Gamma}{\delta L_a} \frac{\delta \Gamma}{\delta \eta_a} &= 0, \\ \left\{ \partial_t^a \frac{\delta}{\delta K_t} + c_\alpha^a \frac{\delta}{\delta K_\alpha} - \frac{\delta}{\delta \xi_a} \right\} \Gamma &= 0, \end{aligned}$$

invariant, and shift the  $s_\alpha^r$  fields by

$$s_\alpha^r = s_\alpha^r - v_\alpha(\epsilon),$$

and choose

$$\begin{aligned} z_n = -\left( \frac{1}{2}\alpha_1 + \beta_1 \right), & \quad \tilde{z}_n = \frac{1}{2}(\beta_1 + \gamma), \\ x_n - \tilde{z}_n - \frac{1}{2}z_n = \frac{1}{2}\alpha_1, & \quad (z_s)_n = -\left( \frac{1}{2}\alpha_2 + \beta_2 \right). \end{aligned}$$

An important lesson to be learned here is that in a general linear gauge, scalar fields can develop gauge-dependent vacuum expectation values, which are innocuous from the renormalization point of view.

### 6.2. Spontaneously broken gauge symmetry

Let us consider the case where  $M^{\dagger}$  is not positive semi-definite. It is by now well known that under such circumstances spontaneous breakdown of the gauge symmetry takes place, and some of the scalar fields and some of the (transverse) gauge bosons combine to form massive vector bosons. We will give here a very brief discussion of the Higgs phenomenon.

We define  $V_0$  by

$$V_0 + \frac{1}{2}(\delta M)_{\alpha\beta} s_{\alpha} s_{\beta} = V. \quad (6.6)$$

If  $M^{\dagger}$  is not positive semi-definite,  $s_{\alpha} = 0$  is no longer a minimum of the potential  $V_0$ . Let  $s_{\alpha} = u_{\alpha}$  be the absolute minimum of  $V_0$ ,

$$\left. \frac{\delta V_0}{\delta s_{\alpha}} \right|_{s_{\alpha}=u_{\alpha}} = 0, \quad (6.7)$$

$$\left. \frac{\delta^2 V_0}{\delta s_{\alpha} \delta s_{\beta}} \right|_{s=u} \equiv \mathcal{M}_{\alpha\beta}^2, \quad \mathcal{M}_{\alpha\beta} \text{ positive definite.} \quad (6.8)$$

$G$ -invariance of the potential  $V_0$  is expressed as

$$s_{\alpha} t_{\alpha\beta}^a \frac{\delta V_0}{\delta s_{\beta}} = 0.$$

Differentiating this with respect to  $s_{\gamma}$  and setting  $s = u$ , and making use of eqs. (6.7), (6.8), we obtain

$$\mathcal{M}_{\alpha\beta}^2 t_{\beta\gamma}^a u_{\gamma} = 0. \quad (6.9)$$

Therefore, there are as many eigenvectors corresponding to the eigenvalue zero as there are linearly independent vectors of the form  $t_{\beta\gamma}^a u_{\gamma}$ . If the dimension of  $G$  is  $N$  and the little group  $g$  which leaves  $u$  invariant has dimension  $m$ , there are  $N - m$  eigenvalues of  $\mathcal{M}^2$  which vanish.

For future use, it is useful to define a vector  $l_{\alpha}^a$  by

$$l_\alpha^a = g^r t_{\alpha\beta}^a u_\beta^r,$$

where  $u_\beta^r$  is to be defined. Since all representations, except the identity representation, of a Lie group are faithful, there are  $N - m$  independent vectors of this form. Now we define

$$(\mu_r^2)_{ab} = l_\alpha^a l_\alpha^b. \tag{6.10}$$

This matrix is of a block diagonal form; moreover if  $a$  or  $b$  refer to a generator of the little group  $g$ ,  $(\mu_r^2)_{ab}$  vanishes. Now form

$$P_{\alpha\beta} = l_\alpha^a (1/\mu_r^2)_{ab} l_\beta^b. \tag{6.11}$$

This is a projection operator,  $P^2 = P$ , onto the vector space spanned by vectors of the form  $t_{\alpha\beta}^a u_\beta^r$ . This space is  $N - m$  dimensional,

$$\text{tr } P = N - m.$$

Eq. (6.9) may be written as

$$\mathcal{M}_{\alpha\beta}^2 P_{\beta\gamma} = 0.$$

We renormalize the gauge fields, ghost fields and gauge coupling constant as before, and renormalize  $s_\alpha$  according to

$$s_\alpha = Z_\alpha^{1/2} (s_\alpha^r + u_\alpha^r + \delta u_\alpha),$$

and determine  $\delta u_\alpha = (\delta u_\alpha)_1 + (\delta u_\alpha)_2 + \dots$  by the condition that the divergences of the form  $-(\delta S[\phi]/\delta s_\alpha) \lambda(\epsilon) v_\alpha$  in the  $n$ -loop approximation be cancelled by the displacement of the renormalized fields  $s_\alpha^r$ ,  $(\delta u_\alpha)_n$ . (See the discussion in subject. 6.1.) The renormalized vacuum expectation value  $u_\alpha^r$  is to be determined by the condition

$$\left. \frac{\delta \Gamma^r}{\delta \phi_i^r} \right|_{\phi_i^r = u_i^r} = 0,$$

$$\left. \frac{\delta^2 \Gamma^r}{\delta \phi_i^r \delta \phi_j^r} \right|_{\phi^r = u^r} \quad \text{positive semi-definite.}$$

We equate the gauge fixing term  $C_\alpha^a$  to (numerically)

$$C_\alpha^a = \alpha^r l_\alpha^a.$$

Then the terms in the action quadratic in renormalized fields and coupling constants (excluding renormalization counter terms) are

$$\begin{aligned} & \{S_{\text{eff}}[\phi^r, \eta^r, \xi^r, g^r]\}_2 \\ &= \int d^4x \left\{ -\frac{1}{2} A_\mu^a(x) \left[ \left( -g^{\mu\nu} \partial^2 + \partial^\mu \partial^\nu \left( 1 - \frac{1}{\alpha} \right) \right) \delta_{a,b} - g^{\mu\nu} \mu_{ab}^2 \right] A_\nu^b(x) \right. \\ & \quad \left. + \frac{1}{2} s'_\alpha \left[ -\partial^2 \delta_{\alpha\beta} - m_{\alpha\beta}^2 - \alpha l_\alpha^a l_\beta^b \right] s'_\beta + \xi_\alpha \left( -\partial^2 \delta_{\alpha b} - \alpha \mu_{ab}^2 \right) \eta_b \right\}, \end{aligned}$$

where we have suppressed the superscript  $r$  altogether. The propagators for the gauge bosons, scalars and ghost fields are, respectively,

$$[\Delta_F^{\mu\nu}(k, \alpha)]_{ab} = - \left( g^{\mu\nu} - \frac{k^\mu k^\nu (1 - \alpha)}{k^2 - \alpha \mu^2 + i\epsilon} \right)_{ac} \left( \frac{1}{k^2 - \mu^2 + i\epsilon} \right)_{cb},$$

which can be written, in a representation in which  $\mu^2$  is block diagonal, as

$$= - \left( g^{\mu\nu} - \frac{k^\mu k^\nu}{\mu^2} \right)_{ac} \left( \frac{1}{k^2 - \mu^2 + i\epsilon} \right)_{cb} + \left( \frac{k^\mu k^\nu}{\mu^2} \right)_{ac} \left( \frac{1}{k^2 - \alpha \mu^2 + i\epsilon} \right)_{cb}$$

or

$$- \left( g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} (1 - \alpha) \right) \frac{1}{k^2 + i\epsilon}, \quad (6.12a)$$

the latter holding for  $a, b$  being one of the  $m$  indices corresponding to generators of the little group  $g$ ;

$$\begin{aligned} [\Delta_F(k^2, \alpha)]_{\alpha\beta} &= (1 - P)_{\alpha\gamma} \left( \frac{1}{k^2 - \alpha \mu^2 + i\epsilon} \right)_{\gamma\beta} \\ & \quad + l_\alpha^a \left( \frac{1}{\mu^2} \right)_{ab} \left( \frac{1}{k^2 - \mu^2 + i\epsilon} \right)_{bc} l_\beta^c, \end{aligned} \quad (6.12b)$$

$$[\Delta_F(k^2, \alpha)]_{ab} = \left( \frac{1}{k^2 - \alpha\mu^2 + i\epsilon} \right)_{ab} \quad (6.12c)$$

If the theory is to be sensible, and gauge invariant, then the poles whose locations depend on the gauge parameter  $\alpha$  cannot be physical, and the particles corresponding to such poles must decouple from the  $S$ -matrix. If this is the case, as we shall show, then there are  $N - m$  massive vector bosons,  $m$  massless gauge bosons, and  $(N - m)$  less scalar bosons than we started out with. This is the Higgs phenomenon.

Is a theory of this kind renormalizable? The answer is yes, because the Feynman rules of the theory, including the propagators above are those of a renormalizable theory, and the WT identity, eq. (6.3), and the ensuing discussion in chapter 5 and subsect. 6.1 hold true whether or not  $M_r^2$  is positive definite. That is, by the methods discussed, we can construct a finite  $\Gamma^r$  in this case also. The expansion coefficients of  $\Gamma^r$  about  $\phi_i^r = u_i^r$ , where

$$\left. \frac{\delta \Gamma^r}{\delta \phi_i^r} \right|_{\phi^r = u^r} = 0,$$

$$\left. \frac{\delta^2 \Gamma^r}{\delta \phi_i^r \delta \phi_j^r} \right|_{\phi^r = u^r} \quad \text{positive definite,}$$

then are the reducible vertices of the renormalized theory. I shall not describe the details of the renormalization program since they have been described in many papers, most recently in my paper (ref. [50]), but the principle involved should be clear.

But an additional remark is in order: the divergent parts of various wave-function and coupling-constant renormalization constants are independent of  $M_r^2$ . This has to do with the fact that these constants are at most logarithmically divergent, and insertion of the scalar mass operators (whose dimension is 2) renders them finite. For detailed arguments, see ref. [50].

### 6.3. Gauge independence of the $S$ -matrix

What remains to be done is to demonstrate that the unphysical poles in the propagators in eq. (6.12), which depend on the parameter  $\alpha$  and some of which correspond to negative metric particles, do not cause unwanted singularities in the renormalized  $S$ -matrix. We shall do this by proving that the renormalized  $S$ -matrix is independent of the gauge fixing parameter  $\alpha$ . To ensure that the  $S$ -

matrix is well-defined, I shall assume that after the spontaneous breakdown of gauge symmetry there is at most one massless gauge boson in the theory.

Before proceeding to the proof, the following illustration is useful. For simplicity let us consider a  $\lambda\phi^4$  theory. The generating functional of Green functions is

$$Z_0[J] = N \int [d\phi] \exp \{iS[\phi] + iJ\phi\}, \quad (6.13)$$

where

$$S[\phi] + J\phi = \int d^4x \left\{ \frac{1}{2} [(\partial\phi)^2 - \mu^2\phi^2] - \frac{1}{4}\lambda\phi^4 + J\phi \right\}. \quad (6.14)$$

What happens if we instead couple the external source to  $\phi + \phi^3$ ? We can write the generating functional as

$$Z[j] = N \int [d\phi] \exp \{iS[\phi] + ij(\phi + \phi^3)\}. \quad (6.15)$$

We can express  $Z$  in terms of  $Z_0$ ,

$$Z[j] = \exp \left\{ i \int d^4x j(x) F \left( \frac{1}{i} \frac{\delta}{\delta J} \right) \right\} Z_0[J], \quad (6.16)$$

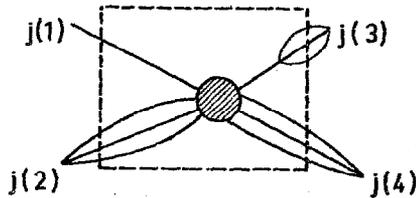
where

$$F(\phi) = \phi + \phi^3.$$

Let us consider a four-point function generated by  $Z[j]$ ,

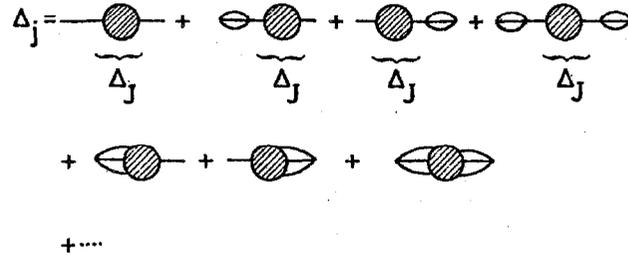
$$G_j[1,2,3,4] = (-i)^4 \frac{\delta^4 Z[j]}{\delta j(1)\delta j(2)\delta j(3)\delta j(4)}.$$

What eq. (6.16) tells us may be pictured as follows:



where we have shown but a class of diagrams that emerge in the expansion of the right-hand side of eq. (6.16). The part of the diagram enclosed by a dotted

square is a Green function generated by  $Z_0[J]$ . Let us now consider the two-point functions  $\Delta_j$  and  $\Delta_J$  generated by  $Z[j]$  and  $Z_0[J]$ ,



So, if we examine the propagators near  $p^2 = \mu_r^2$ , we find

$$\lim_{p^2 \rightarrow \mu_r^2} \Delta_j = \frac{Z_j}{p^2 - \mu_r^2}, \quad \lim_{p^2 \rightarrow \mu_r^2} \Delta_J = \frac{Z_J}{p^2 - \mu_r^2}, \quad (6.17)$$

where the ratio

$$\sigma \equiv (Z_j/Z_J)^{1/2} \quad (6.18)$$

is given diagrammatically by

$$\sigma = 1 + \text{diagram} + \text{diagram} + \dots$$

The diagrammatic representation of  $\sigma$  is shown as a series of diagrams: a horizontal line, a horizontal line with a loop, a horizontal line with two loops, and so on, all connected by plus signs and followed by '+ ...'.

The renormalized  $S$ -matrix is defined by

$$S^r(k_1, \dots) = \prod_{i=1} \lim_{k_i^2 \rightarrow \mu_r^2} \frac{k_i^2 - \mu_r^2}{Z^{1/2}} \tilde{G}(k_1, \dots), \quad (6.19)$$

where  $\tilde{G}$  is the momentum space Green function. Let us consider the unrenormalized  $S$ -matrix defined from  $\tilde{G}_j$ ,

$$S_j^u(k_1, \dots) = \prod \lim(k_i^2 - \mu_r^2) \tilde{G}_j(k_1, \dots). \quad (6.20)$$

Clearly only those diagrams of  $\tilde{G}_j$  in which there are poles in all momentum variables at  $\mu_r^2$  will survive the amputation process. (In fig. 6.1, there are poles in  $k_1^2$  and  $k_3^2$  at  $\mu_r^2$ .) Thus

$$S_j^u(k_1, \dots) = \sigma^{N/2} S_J(k_1, \dots), \quad (6.21)$$

where  $N$  is the number of the external particles. It follows from eqs. (6.18)–(6.21) that

$$S_j^I = S_j^I \equiv S^I, \quad (6.22)$$

and we reach an important conclusion: if two  $Z$ 's differ only in the external source term, both of them yield the same renormalized  $S$ -matrix.

We now come back to the original problem, and ask what happens to  $Z_F[J]$  if an infinitesimal change is made in  $F$ ,

$$\begin{aligned} Z_{F+\Delta F}[J] &= N \int [d\phi d\xi d\eta] \\ &\times \exp i \left\{ S[\phi] + \xi_a \left( \frac{\delta F^a}{\delta \phi_i} + \frac{\delta}{\delta \phi_i} \Delta F^a \right) D_i^b \eta_b - \frac{1}{2} F_a^2 - F_a \Delta F_a + J_i \phi_i \right\}. \end{aligned} \quad (6.23)$$

We are dealing with unrenormalized but dimensionally regularized quantities in eq. (6.23). To first order in  $\Delta F$ , we have

$$\begin{aligned} &Z_{F+\Delta F}[J] - Z_F[J] \\ &= iN \int [d\phi d\xi d\eta] \left\{ -F_a \Delta F_a + \xi_a \frac{\delta \Delta F^a}{\delta \phi_i} D_i^b \eta_b \right\} \\ &\times \exp \{ iS_{\text{eff}}[\phi, \xi, \eta] + iJ_i \phi_i \}. \end{aligned} \quad (6.24)$$

Now, making use of the WT identity (5.23),

$$\int [d\phi d\xi d\eta] \{ F_a - iJ_i \xi_a D_i^b [\phi] \eta_b \} \exp \{ iS_{\text{eff}} + iJ_i \phi_i \} = 0,$$

we can write eq. (6.24) as

$$\begin{aligned} Z_{F+\Delta F} - Z_F &= iN \int [d\phi d\xi d\eta] \left\{ -\Delta F_a \left[ \frac{1}{i} \frac{\delta}{\delta J} \right] \left[ iJ_i \xi_a D_i^b [\phi] \eta_b \right] \right. \\ &\left. + \xi_a \frac{\delta \Delta F^a}{\delta \phi_i} D_i^b [\phi] \eta_b \right\} \exp \{ iS_{\text{eff}} + iJ_i \phi_i \}. \end{aligned}$$

Since

$$-\Delta F_a \left[ \frac{1}{i} \frac{\delta}{\delta J} \right] iJ_i = -\frac{\delta \Delta F_a}{\delta \phi_i},$$

we obtain

$$Z_{F+\Delta F} - Z_F = iJ_i N \int [d\xi d\eta d\phi] \exp \{iS_{\text{eff}} + iJ_i \phi_i\} \\ \times \{-i\Delta F_a[\phi] \xi_a D_i^b[\phi] \eta_b\}. \quad (6.25)$$

But, eq. (6.25) means that to lowest order in  $\Delta F$ ,

$$Z_{F+\Delta F} = N \int [d\phi d\xi d\eta] \exp \{iS_{\text{eff}} + iJ_i \Phi_i\}, \quad (6.26)$$

where

$$\Phi_i = \phi_i - i\Delta F_a[\phi] \xi_a D_i^b[\phi] \eta_b$$

Thus, *an infinitesimal change in the gauge condition corresponds to changing the source term by an infinitesimal amount.* But we have already shown that the renormalized  $S$ -matrix is invariant under such a change! Thus

$$[S^r]_{F+\Delta F} = [S^r]_F. \quad (6.27)$$

A few final remarks: (i) In the previous lectures when we discussed renormalization, we defined the renormalization constants in respect to their divergent parts. The wave-function renormalization constants used in this lecture are defined by the on-shell condition (6.17). These two are related to each other by a finite multiplicative factor. To see this, observe that we can make the propagators finite by the renormalization counter terms defined in the previous lectures. The propagators so renormalized do not in general satisfy the on-shell condition

$$\lim_{p^2 \rightarrow \mu_r^2} \Delta'_F(p^2) = \frac{1}{p^2 - \mu_r^2},$$

but a finite, final renormalization suffices to make them do so. (ii) We can define the coupling constants to be the value of a relevant vertex when all physical external lines are on mass shell. Then

$$\left(\frac{g_r}{g}\right)_{F+\Delta F} = \left(\frac{g_r}{g}\right)_F.$$

## References

*Lecture 1*

For the general discussion of path integral formalism applied to gauge theories, see Prof. Faddeev's lectures, and

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Original literature on the quantization of gauge fields includes

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*Lecture 2*

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*Lecture 3*

For renormalization theory see:

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- [18] S. Weinberg, *Phys. Rev.* 118 (1960) 838.
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- [21] W. Zimmermann, Lectures on elementary particles and quantum field theory, ed. S. Deser, M. Grisaru and H. Pendleton (MIT Press, Cambridge, 1970) p. 395.

The dimensional regularization, in the form discussed here, is due to:

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 [24] G. Cicuta and E. Montaldi, Nuovo Cimento Letters 4 (1972) 329.

A closely related regularization method (analytic regularization) is discussed in:

- [25] E.R. Speer, Generalized Feynman amplitudes (Princeton Univ. Press, Princeton, 1969).

Excellent reviews on the Adler-Bell-Jackiw anomalies are:

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 [27] R. Jackiw, Lectures on current algebra and its applications (Princeton Univ. Press, Princeton, 1970).

For a complete list of anomaly vertices, involving only currents (not pions)

see:

- [28] W.A. Bardeen, Phys. Rev. 184 (1969) 1848.  
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 [31] H. Georgi and S.L. Glashow, Phys. Rev. D6 (1972) 429.  
 [32] C. Bouchiat, J. Iliopoulos and P. Meyer, Phys. Letters 38B (1972) 519.  
 [33] W.A. Bardeen, Proc. 16th Conf. on high-energy physics, 1972, vol. 2, p. 295.

The last reference gives a concise algorithm for dimensional regularization valid for scalar loops. Our prescription agrees with it for this case.

For an excellent discussion of dimensional regularization and a diagrammatic discussion of the Ward-Takahashi identities, see:

- [34] G. 't Hooft and M.J.G. Veltman, Diagrammar, CERN yellow preprint 1973.

#### Lecture 4

The Ward-Takahashi identities were first discussed in the context of quantum electrodynamics in

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