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CLASSICAL RADIATION ZEROS IN GAUGE THEORY AMPLITUDES

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ABSTRACT

The electromagnetic radiation from classical convection currents in relativistic n-particle collisions is shown to vanish in certain kinematical zones, due to the complete destructive interference of the classical radiation patterns of the incoming and outgoing charged lines. We prove that quantum tree photon amplitudes vanish in the same zones, at arbitrary photon momenta, including spin, seagull, and internal line currents, provided only that the electromagnetic couplings and any other derivative couplings are as prescribed by renormalizable local gauge theory (spins ≤ 1). In particular, the existence of this new class of amplitude zeros requires the familiar gyromagnetic ratio value, $g = 2$, for all particles. The location of the zeros is spin independent, depending only on the charges and momenta of the external particles. Such null zones are the relativistic generalization of the well-known absence of electric and magnetic dipole radiation for nonrelativistic collisions involving particles with the same charge-to-mass ratio and g-factor. The origin of zeros in reactions like $u\,\bar{d} \to W^{+}\gamma$ is thus explained and examples with more particles are discussed. Conditions for the null zones to lie in physical regions are established. A new radiation representation, with the zeros manifest and of practical utility independently of whether the null zones are in physical regions is derived for the complete single-photon amplitude in tree approximation, using a gauge-invariant vertex expansion stemming from new internal-radiation decomposition identities. The question of whether amplitudes with closed loops can vanish in null zones is addressed. A low-energy theorem for general quantum amplitudes (including closed loops) is found. Important relations between the photon couplings and Poincaré transformations are discovered. The null zone and these relations are discussed in terms of the Bargmann-Michel-Telegdi equation. The extension from photons to general massless gauge bosons is carried out.

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I. INTRODUCTION

In this paper we describe a new general feature of gauge theories that incorporate massless gauge fields: The existence of zones of null radiation independent of spin.[1] Such null zones should be distinguished from zeros in scattering amplitudes which are imposed, for example, by angular momentum conservation. Here we present a theorem for a new type of zero that can occur in gauge-theory tree-graph amplitudes for photon production/absorption involving any number of spin-0, spin-$\frac{1}{2}$, or spin-1 particles in collision.[1] The theorem, called the radiation interference theorem, can be generalized to other massless gauge bosons.

We find that the kinematic condition for the null radiation zones is simply that all particles must have the same $Q_i/p_i \cdot q$ ratio,

$$\frac{Q_i}{p_i \cdot q} = \frac{Q_j}{p_j \cdot q} \quad , \qquad \text{all } i,j \quad , \tag{1.1}$$

with q the photon momentum. Therefore this condition depends only on the charges Q_i and momenta p_i, q of the external particles in the general scattering process

$$k \text{ particles} \to n\text{-}k \text{ particles} + \text{photon} \quad , \tag{1.2}$$

where, for definiteness, we refer to photon emission. We note that the photon may alternatively appear in the initial state, and in either case (1.1) reduces to $n-2$ independent equations because of charge and momentum conservation.

Under certain restrictions on the couplings, we demonstrate that each helicity amplitude, computed from tree graphs, vanishes for the kinematic null zones defined by (1.1). The restrictions on the couplings, described in detail in the next section, require any derivative couplings to be of gauge theory form. Examples with the prescribed couplings are readily found in which the null zone condition is satisfied in the physical region: (1.1) consistent with four-momentum conservation

$$\sum_{1}^{k} p_i = \sum_{k+1}^{n} p_i + q \quad , \tag{1.3}$$

and the mass shell constraints

$$p_i^2 = m_i^2 \quad , \tag{1.4a}$$

$$q^2 = 0 \quad , \tag{1.4b}$$

as well as charge conservation

$$\sum_{1}^{k} Q_i = \sum_{k+1}^{n} Q_i \quad . \tag{1.5}$$

As a corollary to the theorem, each helicity amplitude can be written as a sum linear in the $n-2$ differences,

$$\Delta_{ij}(Q) \equiv \frac{Q_i}{p_i \cdot q} - \frac{Q_j}{p_j \cdot q} \quad . \tag{1.6}$$

This result is important since it defines a new canonical form (see Sec. VI) for radiation amplitudes and since it does not depend on whether or not the null zone lies in the physical region.

The physical basis of the theorem lies in a corresponding result for classical radiation patterns. Namely, we find that, given (1.1), there is complete destructive interference of the radiation from classical convection currents in relativistic n-particle collisions. In order to see this, we note that the classical amplitude for radiation (frequency ω , direction $\hat{n}$, polarization $\vec{\varepsilon}$) from the relativistic external-line currents (particles with velocities $\vec{v}_i$) reduces to

$$A_{IR} = \frac{1}{\omega}\left(\sum_1^k - \sum_{k+1}^n\right) \frac{Q_i \vec{v}_i \cdot \vec{\varepsilon}}{1-\hat{n}\cdot\vec{v}_i} \tag{1.7}$$

in the low-frequency (infrared = IR) limit. Using a four-vector notation, we can rewrite (1.7) as

$$A_{IR} = \left(\sum_{k+1}^n - \sum_1^k\right) \frac{Q_i}{p_i \cdot q}\, p_i \cdot \varepsilon \quad . \tag{1.8}$$

From momentum conservation (1.3) and transversality

$$q \cdot \varepsilon = 0 \quad , \tag{1.9}$$

it follows from (1.8) that $A_{IR} = 0$ under (1.1).

In the nonrelativistic limit the null zone condition (1.1) reduces to

$$\frac{Q_i}{m_i} = \frac{Q_j}{m_j} \quad , \qquad \text{all } i,j \quad . \tag{1.10}$$

Thus, the zeros of (1.8) can be recognized as the relativistic generalization of the well-known absence of electric dipole radiation for nonrelativistic collisions involving particles with the same charge-to-mass ratio. The classical underpinnings are given in more detail in Sec. III.

Eq. (1.8) and thus the null zone condition directly apply to the simple quantum tree (single-photon) amplitude where all the other particles are spinless and scatter at a point.[2] Moreover, the result is <u>not</u> restricted to low-energy photons. What is surprising about the radiation interference theorem is that it continues to hold when we go onto the next steps of adding contributions from spin currents, gauge-theoretic derivative couplings, and (for $n \geq 4$) internal-line emission in tree approximation.

The restrictions on the interactions specifically require that all photon couplings to the particles correspond to the same gyromagnetic ratio, $g = 2$. In particular, we find that all spin currents can be described by the same first-order Lorentz transformation, a fact that is instrumental in the proof of the radiation interference theorem, but that this description and the null zones are destroyed by anomalous moments ($g \neq 2$) . The equivalence of spin and Larmor precession frequencies is thus intimately related to the null zone phenomena.

Under such gauge-theoretic conditions only the quantum corrections from closed-loop graphs (including anomalous-moment terms) undo the result. Quantum fluctuations in the sources of radiation, required by the uncertainty principle, spoil the exact cancellation; we need the long-range classical currents and perfect plane wave states, such that the particle positions are completely unspecified, for null zones.

The reactions in which a weak boson and a photon are produced by the annihilation of quarks,[3]

$$\begin{aligned} u + \bar{d} &\to W^{+} + \gamma \quad , \\ \bar{u} + d &\to W^{-} + \gamma \quad , \end{aligned} \tag{1.11}$$

measurable in high energy $p\bar{p}$ collisions and which may be important in the verification of the gauge properties of the W, offer striking examples of null radiation zone phenomena. The lowest-order unpolarized cross sections are seen to vanish at an angle unrelated to any angular momentum constraint or specific helicity state.[4] With the neglect of fermion masses, the angular zeros for (1.11) occur at the c.m. angles[4]

$$\cos\theta^{\gamma,n} = \cos\theta^{\gamma,u} = -\frac{1}{3} \tag{1.12}$$

and are indeed destroyed if $g_W \neq 2$. Another example is the reaction[3]

$$\bar{\nu}_e + e^- \to W^- + \gamma \quad , \tag{1.13}$$

where the zero occurs for

$$\cos\theta^{\gamma,\bar{\nu}} = +1 \quad . \tag{1.14}$$

The zeros in the cross sections for (1.11) and (1.13) necessarily imply that each helicity amplitude calculated from the set of four-body analytic tree graphs must have an overall factor $z = \cos\theta - \cos\theta_0$. The interesting algebra which shows this factorization has been developed by Goebel, Halzen and Leveille.[5] Zeros and factorization in other 4-body tree amplitudes have also been discussed in Ref. 5 and by Dongpei.[6] Related work by Grose and Mikaelian concerns the radiative W-decay channels[7] that are the crossed reactions to (1.11) and (1.13). These examples are restricted to $n = 3$, in our notation, where no internal-line photon coupling occurs.

Our motivation for the study of radiation amplitudes stems from the fact that no explanation was known for the $n = 3$ zeros. We now recognize (1.11) and (1.13) as examples of a general class of gauge-theoretic single-photon tree amplitudes that vanish under (1.1), and that are the relativistic generalization of the absence of electric and magnetic dipole radiation for nonrelativistic collisions of particles with the same charge-to-mass ratio and g-factor.[1]

The plan of this paper is as follows: The radiation interference theorem and its corollaries are presented in Sec. II. Developed in Sec. III is the classical basis for the theorem. The conditions under which the null zones lie in physical regions and examples are discussed in Sec. IV and in the Appendix. The detailed proof of the theorem comprises Sec. V.

In the proof, scalar particles with constant couplings are considered first. New decomposition identities for the radiation by an internal line lead to a manifestly gauge-invariant vertex decomposition of the total amplitude. After Dirac and vector particles are added, it is shown how $g = 2$ plays a vital role. Derivative couplings are then taken up and a detailed example follows.

Sec. VI contains a derivation of the radiation representation in terms of the differences (1.6). The special case where some of the particles are neutral is analyzed in Sec. VII. The union of the radiation interference theorem and the standard low-energy theorem for general amplitudes including closed loops is considered in Sec. VIII.

Lorentz invariance plays a fundamental role in the proof of the theorem; this role and the classical Bargmann-Michel-Telegdi (BMT) equations are investigated in Sec. IX. In Sec. X, we show how our analysis can be applied to other gauge groups and to the radiation of other massless gauge bosons. The last section is devoted to a summary and further remarks.

II. THEOREM AND REPRESENTATION FOR RADIATION IN GAUGE THEORIES

The principal result of this paper is a radiation interference theorem, and this section contains its precise statement, a brief outline of its proof, and some implications (corollaries). The details of the proof are given in Sec. V.

We need the following definitions:

1) Gauge-theoretic vertices: We define these to be Lorentz-invariant local interactions involving any number of scalar, Dirac or vector fields with constant couplings but with no derivatives of Dirac fields and at most single derivatives of scalar and vector fields-all of which are aspects of local gauge theories. Products of single derivatives of distinct scalar fields are allowed. All vector derivative couplings must be of the Yang-Mills type; products of such trilinear couplings are also allowed. In particular, the photon couplings must correspond to gyromagnetic ratio $g = 2$ for all spinning particles. Thus such vertices include all renormalizable theories of current physical interest as well as an infinite class of nonrenormalizable theories corresponding to unrestricted numbers of fields.

2) Source graph: This is defined to be any Feynman diagram and serves as a source for photons. Its external lines are labeled by particle four-momenta p_i, charges Q_i, and masses m_i. The external and internal lines may be scalar, Dirac, or vector particles (spin ≤ 1).

3) Radiation graph: We define this as a graph generated by the attachment to a source graph of a single photon, with momentum q, onto a specific line or, in the case of derivative couplings, onto a vertex (seagulls).

4) Radiation amplitude: This is defined as a complete gauge-invariant sum of all the radiation graphs generated from a given source graph(s).

With these definitions, we state the theorem:

Radiation interference theorem: If $M_\gamma(T_G)$ is the radiation amplitude generated by the tree source graph T_G with gauge-theoretic vertices, then

$$M_\gamma(T_G) = 0 \tag{2.1}$$

provided all ratios $Q_i/p_i \cdot q$ are equal.

Comment: We have already noted that the condition on the ratios, previewed in (1.1), is precisely the same as that for the vanishing of classical radiation from incoming and outgoing charged lines (see Sec. III). It may be rewritten

$$\frac{Q_i}{p_i \cdot q} = \frac{Q_1}{p_1 \cdot q} , \quad i = 2,\ldots,n-1 \quad , \tag{2.2}$$

where we have chosen $i=1$ as the standard ratio and $i=n$ as the ratio determined in the limit by the rest. These equations, and the kinematic region (null radiation zone) implied by them, are analyzed in Sec. IV.

The essentials of the proof are as follows:

Proof outline: The theorem is proven first in the special case where T_G is a single vertex V_G. The corresponding radiation amplitude can be written as

$$M_\gamma(V_G) = \sum_i \frac{Q_i J_i}{p_i \cdot q} \tag{2.3}$$

in terms of J_i, the product of the current for photon emission by the i^{th} leg and the remaining vertex factors. Therefore, if

$$\sum_i J_i = 0 \quad , \tag{2.4}$$

then the theorem follows for $M_\gamma(V_G)$. The derivation of (2.4) for spins ≤ 1 and gauge-theoretic vertices is given in Sec. V and is pivotally related to Poincaré invariance (Sec. IX).

The generalization of the proof to tree graphs with internal lines follows from a novel decomposition of the radiation amplitude into a sum over the source vertices of gauge-invariant terms,

$$M_\gamma(T_G) = \sum M_\gamma(V_G)R(V_G) \quad , \tag{2.5}$$

where $M_\gamma(V_G)$ now includes internal legs but for which (2.3) and (2.4) still hold. The factor $R(V_G)$ denotes the propagators and the other vertices of the corresponding source graph. This radiation vertex expansion is also discussed in more detail in Sec. V.

There are several results ancillary to the theorem:

1) Complementary radiation interference theorem: (2.1) also holds if the ratios $\delta_i J_i/p_i \cdot q$ are all equal. (δ_i is defined below.)

This follows from (2.3) and charge conservation,[8]

$$\sum_i \delta_i Q_i = 0 \quad , \tag{2.6}$$

where

$$\delta_i \equiv \begin{cases} +1 & \text{outgoing} \\ -1 & \text{incoming} \end{cases} \quad . \tag{2.7}$$

In general, these amplitude zeros do not lie in any physical region.

2) Radiation representation: In the case of a source graph with a single vertex, the zeros of the interference theorem and its complement imply the double-difference formula,

$$M_\gamma(V_G) = \sum_{i=2}^{n-1} \delta_i p_i \cdot q \left(\frac{Q_i}{p_i \cdot q} - \frac{Q_1}{p_1 \cdot q}\right)\left(\delta_i \frac{J_i}{p_i \cdot q} - \delta_n \frac{J_n}{p_n \cdot q}\right) \quad . \tag{2.8}$$

The off-shell $M_\gamma(V_G)$ in (2.5) can be expressed in a similar manner. This representation, which is discussed in Sec. VI, is an important restatement of the interference theorems, giving us a new canonical form that is true irrespective of whether or not the zeros lie in a physical region and that can be used to simplify calculations.

3) Low-energy theorem: If $\mathcal{M}_\gamma(S_G)$ is the radiation amplitude corresponding to a general source graph S_G which includes closed loops and arbitrary interactions, and if spinning external particles have $g = 2$, then

$$\mathcal{M}_\gamma(S_G) = M_\gamma(S_G) + R_\gamma(S_G) \quad , \tag{2.9}$$

where

$$R_\gamma(S_G) = O(q) \quad , \tag{2.10}$$

and $M_\gamma(S_G)$ satisfies the interference theorem (and thus possesses a radiation representation). This theorem is discussed in Sec. VIII and is essentially the union of the interference theorem (where there is no restriction a priori on the photon momentum) and the standard low-energy theorem.

The radiation theorem and its corollaries are valid when neutral particles are included, subject to a technical stipulation concerning neutral vector particles (Sec. VII). Also, these results are straightforwardly generalized from photons to other massless gauge bosons (Sec. X).

III. CLASSICAL PRELUDE

In this section we examine classical amplitudes for radiation in a general scattering process. We look for null radiation zones where complete destructive interference takes place and, in Sec. V, it will be seen that these null zones carry over exactly to a very general class of quantum tree amplitudes. Some of the classical results have been previewed in the Introduction.

We will show that null zones are in fact the relativistic generalization of the well-known result[9,10] that classical electric dipole radiation vanishes for nonrelativistic collisions of particles with the same charge-to-mass ratio. To review this result, let the i^{th} particle have charge Q_i (e.g., $Q_i = e > 0$ for a proton), mass m_i, and position $\vec{r}_i(t)$, in terms of which the electric dipole moment is

$$\vec{d} = \sum Q_i \vec{r}_i \quad . \tag{3.1}$$

Suppose now that the charge-to-mass ratios are the same for all particles:

$$\frac{Q_i}{m_i} = \frac{Q_1}{m_1} , \ i = 2,3,\ldots, n-1 \quad . \tag{3.2}$$

[Only $n-2$ equations in (3.2) are independent, according to the conservation of charge and mass. See Sec. IV.] Eqs. (3.1) and (3.2) yield

$$\vec{d} = \frac{Q_1}{m_1} \sum m_i \vec{r}_i \quad , \tag{3.3}$$

so, if there are no external forces,

$$\ddot{\vec{d}} = 0 \quad . \tag{3.4}$$

Therefore, the electric dipole radiation field vanishes identically; there is complete destructive interference at all angles. It is important to recognize that this null field situation is the combined result of translational invariance and the constraint (3.2) on the constituent particles.

The fact that we can include spin currents also has its classical origin: Magnetic dipole radiation vanishes for nonrelativistic collisions at a point, when the orbital angular momentum is neglected and when the particles have the same charge-to-mass ratio and the same gyromagnetic factor.[11] To see this, we note that the magnetic dipole moment is

$$\begin{aligned} \vec{\mu} &= \sum \vec{\mu}_i \ , \\ \vec{\mu}_i &= g_i \frac{Q_i}{2m_i} \vec{S}_i \ , \end{aligned} \tag{3.5}$$

with spin $\vec{S}_i$ for each particle. If all g-factors are the same

$$g_i = g_1, \qquad \text{all} \quad i \ , \tag{3.6}$$

then (3.2) and (3.6) imply

$$\vec{\mu} = g_1 \frac{Q_1}{2m_1} \sum \vec{S}_i \quad . \tag{3.7}$$

Therefore, if there are no external torques which interact with the spin,

$$\dot{\vec{\mu}} = 0 \tag{3.8}$$

and the magnetic dipole radiation field vanishes identically.

The relativistic amplitude for radiation during collisions is found from the classical current[12]

$$\begin{aligned} \vec{j}(\vec{x},t) = {} & \left[\theta(-t) \sum_{i=1}^{k} + \theta(t) \sum_{i=k+1}^{n} \right] Q_i \, \vec{v}_i \, \delta(x - \vec{v}_i t - \vec{r}_i(0)) \\ & + [\text{small-distance, small-time corrections}] \ , \end{aligned} \tag{3.9}$$

where k initial particles scatter into $n-k$ final particles with uniform velocities $\vec{v}_i = \dot{\vec{r}}_i$ before or after the collision. Spin currents are ignored for the time being. Thus the classical amplitude for radiation in the direction $\hat{n}$ and with polarization $\vec{\varepsilon}$ by this current for low frequency ω is[10,12,13]

$$A(k,n) = -\frac{1}{\omega}\sum_1^n \delta_i \frac{Q_1}{1-\hat{n}\,\vec{v}_i}\vec{v}_i\cdot\vec{\varepsilon}\,e^{-i\omega\hat{n}\cdot\vec{r}_i(0)} \tag{3.10}$$

denoting the incoming/outgoing sign change by the device in (2.7). The square of the amplitude, $|A|^2$, gives the photon number spectrum into differential Lorentz-invariant phase space, $(2\pi)^{-3}d^3k/2\omega$, when the connection to quantum mechanics is made.

It is seen from (3.9) and (3.10) that the sudden disappearance/appearance of charges is sufficient to determine the infrared limit, $A\to A_{IR}$ as $\omega\to 0$, where

$$A_{IR}(k,n) = -\sum_1^n \delta_i \frac{Q_i}{\omega(1-\hat{n}\cdot\vec{v}_i)}\vec{v}_i\cdot\vec{\varepsilon} \quad . \tag{3.11}$$

This in turn reduces to the correct nonrelativistic electric dipole amplitude, defined as A_{IR}^{nonrel}. For common charge-mass ratios (3.2), we see that

$$A_{IR}^{nonrel}(k,n) = -\frac{Q_1}{\omega m_1}\vec{\varepsilon}\cdot\sum_1^n \delta_i\, m_i\vec{v}_i = 0 \tag{3.12}$$

by conservation of momentum (no external forces), verifying the conclusion reached earlier in (3.4).

We expect (3.11) to be the infrared factor of the corresponding quantum amplitude. For such a comparison (which will be given at the end of this section) and for the null zone discussion, we rewrite (3.11) in terms of the particle four-momenta $p_i = (\vec{E}_i, p_i)$, $p_i^2 = m_i^2$, the photon four-polarization $\varepsilon = (0,\vec{\varepsilon})$, $\varepsilon^2 = -1$, and the photon momentum $q = \omega(1,\hat{n})$, $q^2 = 0$, obtaining

$$A_{IR}(k,n) = \sum_1^n \frac{Q_i}{p_i \cdot q} \delta_i p_i \cdot \varepsilon \qquad (3.13)$$

using (2.7). Since $\hbar = 1$, the photon number spectrum may be computed through (3.13). For common $Q/p \cdot q$ ratios [one is redundant; see Sec. IV],

$$\frac{Q_i}{p_i \cdot q} = \frac{Q_1}{p_1 \cdot q}, \qquad i = 2,3,\ldots,n-1 \quad , \qquad (3.14)$$

we find

$$A_{IR}(k,n) = \frac{Q_1}{p_1 \cdot q} \sum_1^n \delta_i \; p_i \cdot \varepsilon = 0 \qquad (3.15)$$

by momentum conservation (1.3) and transversality (1.9).

We thus have a relativistic generalization for arbitrary photon momenta of the cancellation of electric dipole radiation. [Foretold in (1.1) and (2.2), (3.14) has already been observed to reduce to (3.2) in the nonrelativistic limit.] Because the fields get folded forward, the general cancellation occurs only for the set of charges and momenta that satisfy (3.14), ranges for which are discussed in Sec. IV. It is (3.14) that reduces to the angular zero in the lowest-order weak-boson amplitude for reactions (1.11) and (1.13) and that is the condition for zeros in the very general class of tree amplitudes defined in Sec. II.

The classical treatment of the radiation generated by a system of moving intrinsic magnetic moments is relatively complicated except in the low-frequency, nonrelativistic limit. In that limit the individual magnetic moments can be represented by their intrinsic (rest frame) values, (3.5) and the corresponding radiation amplitude is[10]

$$A_m = i \sum_1^n \delta_i \; (\vec{\mu}_i \times \hat{n}) \cdot \vec{\varepsilon} \; e^{-i\omega \hat{n} \cdot \vec{r}_i(0)} \quad , \qquad (3.16)$$

noting the absence of ω^{-1} in comparison with (3.10). The expression (3.16) does indeed vanish if the charge-to-mass ratios are all the same, (3.2), if the g-factors are all the same, (3.6), and if the total intrinsic spin is conserved.

Orbital angular momentum, through its associated magnetic moment, contributes terms at the ω^0 level as well. To see this, consider the next leading term in (3.10) and the identity

$$(\hat{n}\cdot\vec{r}_i)\vec{v}_i = \frac{1}{2}(\vec{r}_i \times \vec{v}_i) \times \hat{n} + \frac{1}{2}\left[(\hat{n}\cdot\vec{r}_i)\vec{v}_i + (\hat{n}\cdot\vec{v}_i)\vec{r}_i\right] \quad . \tag{3.17}$$

The antisymmetric term in (3.17) leads to an additional magnetic moment contribution in (3.16) corresponding to the replacement $g_i\vec{S}_i \rightarrow \vec{L}_i + g_i\vec{S}_i$ in (3.5); the symmetric term in (3.17) leads to a quadrupole amplitude.

Rather than proceeding further in a semi-classical manner, we will turn our attention to quantum amplitudes, for which we have already found the infrared factor exactly. To see this for the arbitrary quantum amplitude M shown in Fig. 1a, note that the infrared terms are contained in the graphs where the photon is attached to the external legs, as in Fig. 1b. If the scattering amplitude for k particles → n-k particles is denoted by $T(p_1,\ldots,p_n)$, then the ω^{-1} term is given by[12]

$$\begin{aligned} M_{IR} &= \left[\sum_1^k \frac{Q_i}{(p_i-q)^2 - m_i^2}(2p_i - q)\cdot\varepsilon + \sum_{k+1}^n \frac{Q_i}{(p_i+q)^2 - m_1^2}(2p_i + q)\cdot\varepsilon\right] T(p_1,\ldots,p_n) \\ &= A_{IR}(k,n)T(p_1,\ldots,p_n) \quad , \end{aligned} \tag{3.18}$$

in view of (3.13). Clearly, M_{IR} vanishes when (3.14) is satisfied and, indeed, the radiation interference theorem always holds for the infrared part of any amplitude. Such zeros in the infrared factor have apparently gone unnoticed until now. See Sec. VIII.

IV. Q/p•q FACTORS AND PHYSICAL NULL ZONES

Prior to the proof in the next section of the radiation interference theorem, we examine the implications of the null zone equations. We consider the region in the photon, n-particle phase space where (2.2) satisfied and the question of whether this region is physical or not. Examples and theorems are presented in this section following preliminary definitions, identities, and a demonstration that only n-2 equations are independent.

A. Preliminaries

Definition: The null radiation zone is the momentum-space region where all the $Q/p\cdot q$ factors[14] are equal. The corresponding n-2 equations can be expressed generally as

$$\frac{Q_i}{p_i\cdot q} = \frac{Q_j}{p_j\cdot q}\ , \qquad \text{all} \quad i \neq j,\ell \quad , \tag{4.1}$$

for fixed distinct pairs j,ℓ. [Cf. (2.2).] The (classical) infrared factor (3.13) always vanishes for all radiative reactions in the null radiation zone.

The reason that the n-1 possible equations reduce to n-2 follows from charge and momentum conservation, and the following interesting identity:

$$\frac{a+b+c+\cdots}{A+B+C+\cdots} - \frac{a}{A} = \left[\left(\frac{b}{B} - \frac{a}{A}\right)B + \left(\frac{c}{C} - \frac{a}{A}\right)C + \cdots\right]\frac{1}{A+B+C+\cdots} \quad , \tag{4.2}$$

which is a generalization of

$$\frac{a+b}{A+B} - \frac{a}{A} = \left(\frac{b}{B} - \frac{a}{A}\right)B\,\frac{1}{A+B} \quad . \tag{4.3}$$

As a special case of (4.2), we have

$$\frac{a}{A} = \frac{a+b+c+\cdots}{A+B+C+\cdots} \quad , \tag{4.4a}$$

if

$$\frac{a}{A} = \frac{b}{B} = \frac{c}{C} = \cdots \quad . \tag{4.4b}$$

These relations may be rescaled by $b \to \beta b$, $B \to \beta B$, $c \to \gamma C, \ldots,$ for arbitrary $\beta, \gamma, \ldots$ The cases where we rescale by the value -1 are of particular interest.

Therefore, charge conservation (1.5), momentum conservation (1.3), the masslessness of the photon (1.4b), and (4.4) imply that

$$\frac{Q_\ell}{p_\ell \cdot q} = \frac{Q_j}{p_j \cdot q} \quad , \tag{4.5}$$

if (4.1) holds. The last $Q/p \cdot q$ factor is determined by the rest through (4.2) and all n must be equal if $n-1$ of them are equal.

A caveat exists for any attempt to use an arbitrary set of $n-2$ equalities for the $Q/p \cdot q$ factors in place of (4.1), since they may not always be independent. For example, the electron-electron reaction,

$$e^-(p_1) + e^-(p_2) \to e^-(p_3) + e^-(p_4) + \gamma(q) \quad , \tag{4.6}$$

has a null zone given by $p_1 \cdot q = p_2 \cdot q = p_3 \cdot q = p_4 \cdot q$. But $p_1 \cdot q = p_3 \cdot q$ is is equivalent to $p_2 \cdot q = p_4 \cdot q$ by momentum conservation and, therefore, they are not independent equations. This problem does not arise if the prescription in (4.1) is followed.

B. Null Zone: General Remarks

Since $p_i \cdot q$ is positive semi-definite, the first restriction from (4.1) is that all nonzero charges in both the initial and final state must have the same sign,

$$\frac{Q_i}{Q_j} \geq 0 \,, \qquad \text{all} \quad i,j \quad . \tag{4.7}$$

This includes neutral particles which are required by the null zone condition to have zero mass and to travel in the same direction as the photon. (Neutral particles are addressed in more detail in Sec. VII.) For a given initial (final) state, the more final (initial) particles there are, the smaller their charges, and consequently fractional charges can play a special role.[15]

In the nonrelativistic limit for all n particles, (4.1) gives the familiar result

$$\frac{Q_i}{m_i} = \frac{Q_j}{m_j} \,, \qquad \text{all} \quad i \neq j,\ell \quad , \tag{4.8}$$

which is equivalent to Eq. (3.2) and is indeed satisfied only for same-sign charges. Note that, in the nonrelativistic limit, mass conservation replaces momentum conservation in justifying the reduction from $n-1$ to $n-2$ equations, again noting (4.4).

Although we shall show that physical null zone configurations do exist in realistic examples, it is emphasized again that the radiation interference theorem goes beyond whether any physical null zone can be found since the radiation representation always holds. For the same reason, the fact that the ratios $Q_i J_i/p_i\cdot q$ have no (nontrivial) identical values does not leave the complementary radiation interference theorem empty of content.

C. Null Zone: $n \leq 3$

Given that all nonzero charges are of the same sign, the next step is to find the null zone constraints on the energies and angles.

1. $\underline{n=1}$ For completeness, we include this "mixing" transition which is only realized off-shell for well-defined particle states and has a tadpole source graph. The radiation representation is trivially zero since $Q_1 = 0$.

2. $\underline{n=2}$ This occurs, for example, in $\mu \to e\gamma$ lepton-number-violating radiative decays. The momentum and charge conservation equations, $p_1 = p_2 + q$ and $Q_1 = Q_2$, respectively, automatically satisfy $Q_1/p_1\cdot q = Q_2/p_2\cdot q$, in accord with the fact that there is no independent equation in (4.1). Thus the radiation representation is identically zero and, indeed, the most general amplitude[16] $\bar{\Psi}(a + B\gamma_5)\sigma_{\mu\nu}\psi\, q^\mu\varepsilon^\nu$ is $O(q)$, with contributions from derivative couplings on closed loops. (See Sec. VIII.)

3. $\underline{n=3 \text{ decay}}$ We consider the decay process where $p_1 = p_2 + p_3 + q$ and $Q_1 = Q_2 + Q_3$. The single null zone equation can be taken as

$$\frac{Q_2}{p_2\cdot q} = \frac{Q_3}{p_3\cdot q} \quad . \tag{4.9}$$

In the rest frame of the parent, take the two free variables to be the energies E_2 and E_3 , or, rather, the variables

$$x \equiv \frac{2p_3 \cdot q}{m_1^2} = 1 - \frac{2E_2}{m_1} + \mu_2^2 - \mu_3^2 \quad ,$$
$$y \equiv \frac{2p_2 \cdot q}{m_1^2} = 1 - \frac{2E_3}{m_1} + \mu_3^2 - \mu_2^2 \quad , \tag{4.10}$$

where

$$\mu_i \equiv m_i/m_1 \quad . \tag{4.11}$$

The variables x,y coincide with those of Ref. 7 in the limit $m_2 = m_3 = 0$. Eq. (4.9) may be rewritten

$$y = \frac{Q_2}{Q_3} x \quad , \tag{4.12}$$

and the question before us is whether this straight line intersects the physical domain in x-y space.

The boundary limits on x and y are derived in the Appendix. The overall x range is

$$0 \le x \le (1 - \mu_2)^2 - \mu_3^2 \quad , \tag{4.13}$$

and, for a given x in (4.13), the y range is

$$y_- \le y \le y_+ \quad ,$$
$$y_\pm \equiv \frac{x}{2A} [B \pm (B^2 - 4\mu_2^2 A)^{\frac{1}{2}}] \quad ,$$
$$A \equiv x + \mu_3^2 \quad , \tag{4.14}$$
$$B \equiv 1 - \mu_2^2 - \mu_3^2 - x \quad .$$

The roles of x and y may be reversed by relabeling $2 \leftrightarrow 3$.

In the massless limit $m_2 = m_3 = 0$, the inequalities (4.13) and (4.14) reduce to the case already discussed in Ref. 7 :

$$\begin{aligned} &0 \leq x \leq 1 \quad , \\ &0 \leq y \leq 1 - x \quad . \end{aligned} \tag{4.15}$$

Thus, there is always a line of intersection in $x - y$ space between (4.12) and (4.15) as long as the three charges have the same sign. In a situation where m_2 and m_3 can be neglected, e.g., W-decay,[7] there is always a physical null line for each $n = 3$ decay helicity amplitude.

In general, the range of values of Q_2/Q_3 for which we have a physical null zone will be limited according to the given set of masses m_2 and m_3 . This is discussed in the Appendix, and we quote one particularly interesting result. Namely, there is a physical null zone for all masses and charges such that $Q_2/m_2 = Q_3/m_3$, $m_2 + m_3 \leq m_1$. This charge-to-mass ratio stipulation is consistent with the soft-photon, nonrelativistic limit, where $m_2 + m_3 = m_1$ and all Q_i/m_i are equal, and is a special case of a general theorem to be presented later.

4. <u>$n = 3$ scattering</u> For the reaction where $p_1 + p_2 = p_3 + q$ and $Q_1 + Q_2 = Q_3$, the single null zone equation can be taken as

$$\frac{Q_1}{p_1 \cdot q} = \frac{Q_2}{p_2 \cdot q} \quad . \tag{4.16}$$

In terms of c.m. variables, the angle θ between $\vec{p}_1$ and $\vec{q}$ is derived from (4.16) to be

$$P\cos\theta = \frac{Q_2 E_1 - Q_1 E_2}{Q_2 + Q_1} \quad , \tag{4.17}$$

with $P \equiv |\vec{p}_1| = |\vec{p}_2|$.

The physical null zone for the $n = 3$ four-body amplitudes corresponds to those angles θ of (4.17) for which $|\cos\theta| \leq 1$, and is discussed for general masses and charges in the Appendix. In the ultrarelativistic limit $(m_1, m_2 \to 0)$, (4.17) yields

$$\cos\theta = \frac{Q_2 - Q_1}{Q_2 + Q_1} \quad , \tag{4.18}$$

and all positive Q_2/Q_1 values produce physical null points. It is seen that (4.18) checks[4,6] with (1.12) and (1.14). The nonrelativistic limit is consistent with unrestricted θ (total interference at all angles). The Appendix contains a demonstration that, if $Q_1/m_1 = Q_2/m_2$, a physical null zone exists whatever the energies, again a special case of a more general theorem.

D. Null Zone: $n = 4$ example

It is now possible to build up the results for larger n from the $n = 3$ analysis. For $n = 4$, consider the $2 \to 3$ process where $p_1 + p_2 = p_3 + p_4 + q$ and $Q_1 + Q_2 = Q_2 + Q_3$. This is equivalent to a three-body decay of a system with mass $E = E_1 + E_2$ (the total c.m. energy). The photon angle is still given by (4.17), using (4.16) as one of the two null zone equations. The second null zone equation is

$$y = \frac{Q_3}{Q_4} x \quad , \tag{4.19}$$

expressed in terms of variables analogous to (4.10),

$$x \equiv \frac{2p_4 \cdot q}{E^2} = 1 - \frac{2E_3}{E} + \frac{m_3^2 - m_4^2}{E^2} \quad ,$$
$$y \equiv \frac{2p_3 \cdot q}{E^2} = 1 - \frac{2E_4}{E} + \frac{m_4^2 - m_3^2}{E^2} \quad . \tag{4.20}$$

The two null zone equations, (4.16) and (4.19), do not follow the prescription of (4.1), nevertheless, they are independent.

We count the dimensions of the null zone by recalling that the photon polar angle is fixed and noting that its azimuth can be arbitrarily chosen. The energy of particle 4 is determined by (4.19). After four-momentum conservation, the last two free dimensions may be taken to be the energy x of particle 3 and the azimuth of the plane of particles 3 and 4 (and γ) relative to the photon axis. These constitute a 2-dimensional null zone.

We may use the previous decay equations in (4.12)-(4.14) and in the Appendix, mutatis mutandis, to determine whether the null zone is in the physical region. In particular, if the ratios Q_i/m_i are all identical (see subsection E), there is a physical null zone for any c.m. energy. This suggests a striking example.

Bremsstrahlung in electron scattering, (4.6), satisfies the radiation theorem in lowest order and, in addition, the Q_i/m_i ratios are identical for all charges. Thus, we discover amplitude zeros in a textbook reaction that have gone unnoticed up to now and that occur somewhere for all energies $(E \geq 2m\,,\, m_i = m)$. Having two (or more) source graphs is immaterial. The physical null zone is the two-dimensional region explained above and in the Appendix:

$$
\begin{aligned}
&E'(1 - v'\cos\theta') = E/2 \quad , \\
&\pi/2 \le \theta' \le \pi \quad , \\
&0 \le \phi' \le 2\pi \quad , \\
&\theta = \pi/2 \quad ,
\end{aligned}
\tag{4.21}
$$

in which $E_3 = E_4 \equiv E'$, the final velocities $v_3 = v_4 \equiv v'$, $\theta_3 = \theta_4 \equiv \theta'$, and the photon energy is $\omega = E - 2E' = -2E'v'\cos\theta'$. The final-state plane of the two electrons and the photon has an azimuthal angle ϕ' about the photon axis, pictured with the other variables in Fig. 2.

In contrast to identical scalar bosons, the radiative region (4.21) shown in Fig. 2 is not forbidden by angular momentum conservation for identical spin - $\frac{1}{2}$ fermions. It is radiation interference, and not the exclusion principle, that leads to a zero in the (tree) radiation amplitude for reaction (4.6). The fact that closed loops can destroy the radiation zero but not angular-momentum zeros provides one test for such interpretations, when the two mechanisms overlap, and is discussed in Sec. VIII.

E. Null Zone: Theorem

As n increases, the null zone analysis becomes increasingly complicated. However, it is possible to give a general criterion for the existence of physical null zones:

Physical null radiation zone theorem: There is a null radiation zone for any c.m. energy in the physical region of the reaction, k particles $\rightarrow$ n-k particles + photon, if the initial particles have an identical charge-to-mass ratio and the final particles share another common charge-to-mass ratio, not necessarily the same as the initial ratio.

Corollary: As a special limit of this theorem, one can require instead that the initial and/or the final particles be massless.

In short, we can always find physical regions where all $Q/p \cdot q$ are equal, provided that the Q/m are equal or that the particles are massless, conditions which can be restricted separately to the initial or final states. In decay processes, obviously, the parent must not be massless, and in all cases the nonzero charges must have the same sign. We note also that, alternatively, the photon may be in the initial state.

The proof and further remarks concerning this theorem, its corollary, and their variations are given in the Appendix. It should be noted that, in the event there are more than two particles in the initial state, a physical null zone corresponds to limited regions of initial as well as final phase space. The point is that such regions can always be found, under the conditions of the theorem. Earlier examples can be compared to the theorem, noting that electron scattering (4.6) and W production (1.11) with massless quarks and arbitrary quark charges conform to the theorem and its corollary, respectively. The corollary is particularly useful for high energy limits.

V. PROOF OF THE THEOREM

The proof of the radiation interference theorem is carried out initially for spinless particles and no derivative couplings. The extension of the proof to encompass spin and gauge theoretic vertices is subsequently taken up and a detailed example is given.

A. Spin-zero fields and constant couplings

We first consider scalar/pseudoscalar particles whose couplings to each other may involve an arbitrary number of fields but no derivatives. Their photon coupling, on the other hand, is the standard convective derivative trilinear interaction.

A vertex source graph, $V_G(n)$, is defined to have n external lines coupled through a single vertex (Fig. 3a). In the absence of derivative couplings in $V_G(n)$, only external-line photon attachments (Fig. 3b) are present in the corresponding radiation amplitude. For photon emission (momentum q, polarization ε) by an external scalar leg with charge Q flowing along momentum p, we have the following (convection current) factors:[13]

$$\text{outgoing particle: } \frac{Q}{p\cdot q}\; p\cdot\varepsilon \quad , \tag{5.1a}$$

$$\text{incoming particle: } (-p\cdot\varepsilon)\,\frac{Q}{p\cdot q} \quad , \tag{5.1b}$$

where p is outgoing [incoming] in (5.1a) [(5.1b)]. We note that Eqs. (5.1) are invariant under $p \to p \pm q$.

Let λ_n , which carries dimension if $n \neq 4$, denote the constant vertex in $V_G(n)$. Then the radiation amplitude is

$$M_\gamma[V_G(n)] = \lambda_n\, A_{IR}\,(k,n) \quad , \tag{5.2}$$

where $A_{IR}(k,n)$ is the classical amplitude, (3.13). Therefore the proof of the theorem in this instance is immediate.

The $n=3$ vertex is the spinless version of (1.11) which is known[5] to have the same amplitude zero. The new aspects of the preceding results for vertex source graphs are the demonstration that amplitude zeros also exist for $n>3$ (an infinite class) together with the identification of the conditions (4.1) for their location, and the understanding of the physical basis for their occurrence (the interference of classical radiation patterns).

To generalize the proof to arbitrary tree graphs, we must take into account the radiative contributions from photon attachments to internal lines. (Cf. Fig. 4.) Remarkably, the same zeros survive. A crucial step in handling such contributions involves the use of an identity for real photon emission from a scalar internal line (mass m, charge Q, and momentum change from p to $p' \equiv p-q$):

$$i\frac{1}{p'^2-m^2}\,Q(p'+p)\cdot\varepsilon\,\frac{1}{p^2-m^2} = \frac{i}{p'^2-m^2}\,\frac{Q}{p'\cdot q}\,p'\cdot\varepsilon + (-p\cdot\varepsilon)\,\frac{Q}{p\cdot q}\,\frac{i}{p^2-m^2}\quad , \qquad (5.3)$$

using $q\cdot\varepsilon = q^2 = 0$.

Eq. (5.3) is the first of a set of new identities to be used for real-photon couplings to internal particles. The spin-½ and spin-1 versions appear in the next two sections, and we refer to these as <u>radiation decomposition identities</u>, since they represent a split of the internal vertex into two terms each of which is a product of a propagator and a quasi-external-leg emission factor. (These external-leg emission factors are called "quasi" since their momenta are off-shell.) This decomposition is graphically illustrated in Fig. 5 and is manifestly gauge invariant.

In the scalar case (5.3) holds to all orders. The invariant-amplitude expansion for the scalar-photon-scalar vertex function, $\Gamma^\mu = (p' - p)^\mu f(p'^2, p^2) + (p' + p)^\mu g(p'^2, p^2)$, implies that $\Gamma \cdot \varepsilon = (p' + p) \cdot \varepsilon \; g$. Alternatively, the Ward-Takahashi identity can be used to show that $\Delta'(p'^2)^{-1} - \Delta'(p^2)^{-1} = -2p \cdot q \; g$, where Δ' is the full scalar propagator. Thus (5.3) is valid with $(p' + p) \cdot \varepsilon$ replaced by $\Gamma \cdot \varepsilon$ and the free propagators replaced by Δ'.[17]

Let us illustrate the use of (5.3) with an $n = 4$ example, photon emission from a t-channel-exchange source graph, depicted in Fig. 6. We find that the five graphs in the radiation amplitude can be expressed in the form

$$M_\gamma(\text{Fig. 6.}) = \frac{i\lambda_3^2}{(p_3 - p_2)^2 - m_5^2} \left[\frac{Q_4}{p_4 \cdot q} p_4 \cdot \varepsilon - \frac{Q_1}{p_1 \cdot q} p_1 \cdot \varepsilon + \frac{Q_1 - Q_4}{(p_1 - p_4) \cdot q} (p_1 - p_4) \cdot \varepsilon \right] + \frac{i\lambda_3^2}{(p_1 - p_4)^2 - m_5^2} \left[\frac{Q_3}{p_3 \cdot q} p_3 \cdot \varepsilon - \frac{Q_2}{p_2 \cdot q} p_2 \cdot \varepsilon + \frac{Q_2 - Q_3}{(p_2 - p_3) \cdot q} (p_2 - p_3) \cdot \varepsilon \right] , \tag{5.4}$$

where $p_1 + p_2 = p_3 + p_4 + q$, $Q_5 = Q_1 - Q_4 = Q_3 - Q_2$. [A radiation representation for (5.4) is given in Sec. VI.]

We notice that the two quantities in square brackets in (5.4) are the classical A_{IR} amplitudes (3.13). These amplitudes are separately gauge invariant, and each is associated with one of the $n = 3$ source vertices. Both quantities are multiplied by the original source graph amplitude, but with different kinematics in the two cases. The momentum assignment in each source amplitude is determined by momentum conservation at the other vertex. These features are quite general and are reflected in the proof to which we return.

From a general scalar tree graph T_G and (5.3), we obtain a radiation vertex expansion [cf. (2.5)]:

$$M_\gamma(T_G) = \sum_v \lambda_v A_{IR}(v)\ R(v) \quad , \tag{5.5}$$

summing over the vertices v of T_G. Here, $A_{IR}(v)$ is the gauge-invariant off-shell version of the classical amplitude (3.13) for radiation by the legs of vertex v and is given by

$$A_{IR}(v) = \sum_i \delta_i \frac{Q_i}{p_i \cdot q} p_i \cdot \varepsilon \quad , \tag{5.6}$$

where the sums are over all external and internal lines into and out of the vertex. Also, $R(v)$ is comprised of the remaining factors in T_G including all propagators and with the momentum assignments consistent with photon momentum q leaving vertex v. [$R(v)$ is simply T_G/λ_v in the scalar case, but with momentum unconserved at the vertex v.]

The validity of (5.5) follows from the fact that (5.3) partitions each internal-line photon attachment into two quasi-external-line attachments which are respectively and unambiguously assigned to the two vertices joined by the internal line. For every vertex v, we are left with a complete set of photon emission factors, one factor for each attached line and each factor with the same coefficient $R(v)$. The momentum of each propagator on the right-hand-side of (5.3) is consistent with q leaving the vertex to which its quasi-external factor is ultimately associated, giving the same R that the external-leg radiation does.

The last step in the proof for scalar tree graphs follows from the fact that through (4.2) the internal $Q/p\cdot q$ factors are determined by the external ones. If (4.1) is satisfied, then all factors are equal, internal and external, so that

$$\frac{Q_I}{p_I \cdot q} = \frac{Q_j}{p_j \cdot q} \tag{5.7}$$

for all I . Therefore, each $A_{IR}(v)$ (and consequently M_γ) vanishes in the null zone.

The theorem can be checked by the $n=4$ example in (5.4), using (5.6). In addition, this example demonstrates the interesting case of vanishing internal charges, which may occur even though all external charges have the same sign. Suppose that $Q_1 = Q_4$ and (so) $Q_2 = Q_3$, leaving $Q_5 = 0$. One null zone condition is $p_1 \cdot q = p_4 \cdot q$ (or equivalently $p_3 \cdot q = p_2 \cdot q$) which is the same as $(p_3 - p_2)^2 = (p_1 - p_4)^2$, and therefore the cancellation still goes through in (5.4) but now <u>between</u> the square brackets. This is not surprising since the original demonstration did not depend on the magnitudes of Q_i, and the limit $Q_1 \to Q_4$ could be taken before or after demanding (4.1). In general, we may regard any two vertices connected by a neutral internal line as a single compound vertex in expansions like (5.5). Neutral external scalar lines conform to the theorem as well but in a more subtle fashion. Their inclusion is analyzed in Sec. VII.

We conclude this subsection with a few general remarks. Recall that, since the null zone cancellation depends only on the values of the external charges and momenta, <u>all</u> source graphs (with the prescribed couplings) generate tree radiation amplitudes that vanish at exactly the same places for a given set of external particles. We also point out that the proof breaks down for closed-loop source graphs, since (5.7) does not follow unless the internal-line momentum is fixed by the external momenta. However, the analysis is applicable to the tree substructure and does imply a cancellation to $O(q)$ for arbitrary amplitudes, according to the discussion in Sec. VIII. Finally, we remark that zeros in tree graphs, with classical propagation between vertices and without spin and derivative couplings, might very well be expected to be "classical" and thus derivable from considerations such as those in Sec. III. Yet, (3.13) is a low-frequency result while the tree graph demonstrations are seen to go through for arbitrary photon momentum.

B. Including spin-half particles

Next we extend the proof of the radiation interference theorem to include Dirac particles. Specifically, each tree source graph may now involve any even number $2D$ of Dirac particles along with an arbitrary number $n-2D$ of scalars (but no derivative couplings). The (only) new ingredient is the Dirac spin current and the strategy of the preceding subsection may be followed.

A vertex source graph may be written,

$$V_G(n,D) = \prod_{i=1}^{D} \bar{w}_i' \Gamma_i w_i \quad , \tag{5.8}$$

in terms of D spin bilinears. (w,w' are chosen as needed from the familiar u,v spinors.) The Γ_i are constant matrices in spin space and, in view of the Lorentz invariance of V_G, it is left understood that they may be summed over in various combinations, as in, for example, $\bar{u}_j'\gamma_\mu u_j \bar{v}_k'\gamma^\mu u_k$. Any coupling factor representing the presence of the $n-2D$

scalars can be absorbed into the Γ_i .

The factors corresponding to (5.1) for photon emission by an external Dirac leg are computed from minimal (gauge theoretic) coupling to be

$$\text{outgoing particle:} \qquad \frac{Q}{p\cdot q}\,\bar{u}(p)(p\cdot\varepsilon + \frac{1}{4}\,[\not\varepsilon,\not q]) \quad , \tag{5.9a}$$

$$\text{incoming particle:} \qquad (-\,p\cdot\varepsilon - \frac{1}{4}\,[\not\varepsilon,\not q])\,u(p)\,\frac{Q}{p\cdot q} \quad , \tag{5.9b}$$

$$\text{outgoing antiparticle:} \qquad (p\cdot\varepsilon - \frac{1}{4}\,[\not\varepsilon,\not q])\,v(p)\,\frac{Q}{p\cdot q} \quad , \tag{5.9c}$$

$$\text{incoming antiparticle:} \qquad \frac{Q}{p\cdot q}\,\bar{v}(p)(-\,p\cdot\varepsilon + \frac{1}{4}\,[\not\varepsilon,\not q]) \quad . \tag{5.9d}$$

We see that each is a sum of convection and spin currents, and each replaces the original spinor in the <u>source</u> graph. The absence of explicit mass dependence in (5.9) is fundamental to the use of minimal coupling and can be compared to Eq. (5.26) below.

The radiation amplitude for the vertex source graph (5.8) can be obtained directly from (5.1) and (5.9). With k initial particles,

$$M_\gamma[V_G(n,D)] = V_G(n,D)\;A_{IR}(k,n) + \sum_{i=1}^{D} S_i \prod_{j=i}^{D} \bar{w}_j'\,\Gamma_j\,w_j \quad , \tag{5.10}$$

where

$$S_i = \frac{1}{4}\,\bar{w}_i'(p_i')\;\{\,\frac{Q_i'}{p_i'\cdot q}\,[\not\varepsilon,\not q]\,\Gamma_i - \Gamma_i\,[\not\varepsilon,\not q]\,\frac{Q_i}{p_i\cdot q}\,\}\;w_i(p) \quad . \tag{5.11}$$

The combined convection currents, one from each leg, give the term in (5.10) with the classical amplitude factor, (3.13), which clearly vanishes in the null zone.

We can show that the Dirac spin currents also conspire to cancel, in the null zone, but by Lorentz invariance rather than by translational invariance. The key to this result is a relationship between Lorentz transformations and the minimal photon-spin-½ coupling. Namely, the spin currents in (5.9) are proportional to first-order wave-function corrections all of which can be associated with the same (called "universal" hereafter) first-order Lorentz transformation,

$$\Lambda_{\mu\nu} = g_{\mu\nu} + \lambda\omega_{\mu\nu} \quad , \tag{5.12}$$

where

$$\omega_{\mu\nu} = q_\mu\varepsilon_\nu - \varepsilon_\mu q_\nu \tag{5.13}$$

and λ is an infinitesimal length. The spinor wave function ψ Lorentz-transforms as[18]

$$\psi'(x') = S(\Lambda)\,\psi(x) \quad , \tag{5.14}$$

where $x' = \Lambda x$ and, in first order,

$$S(\Lambda) = 1 - \frac{i}{4}\,\lambda\sigma_{\mu\nu}\omega^{\mu\nu} = 1 - \frac{1}{4}\,\lambda\,[\not\varepsilon,\not q] \quad , \tag{5.15}$$

using (5.12) and (5.13). Comparison of (5.9) and (5.15) establishes the relationship.

When the $Q/p\cdot q$ factors are equal, (5.10) reduces to

$$M_\gamma[V_G(n,D)] = \frac{Q_1}{p_1\cdot q}\sum_{i=1}^{D} \bar{w}_i'\,\Delta\Gamma_i w_i \prod_{j\neq i}^{D} \bar{w}_j'\,\Gamma_j\, w_j \qquad \text{(null zone)} \quad , \tag{5.16}$$

with

$$\Delta\Gamma_i = \frac{1}{4}\,[\not{\epsilon},\not{q}]\,\Gamma_i - \Gamma_i\,\frac{1}{4}\,[\not{\epsilon},\not{q}] \tag{5.17}$$

$$= \frac{i}{4}\,[\sigma_{\mu\nu}\omega^{\mu\nu},\ \Gamma_i] \quad . \tag{5.18}$$

We see that (5.16) is proportional to the complete first-order change (5.8), under (5.12), since $\lambda\,\bar{w}_i'\,\Delta\Gamma_i\,w_i$ is the first-order change in[19] in $\bar{w}_i'\,\Gamma_i\,w_i$. By the Lorentz invariance of V_G, we conclude that

$$M_\gamma[V_G(n,D)] = 0 \qquad \text{(null zone)} \quad . \tag{5.19}$$

This completes the proof for vertex source graphs.

To extend the proof to an arbitrary tree source graph with internal lines, we need an identity, for a real photon attached to an off-shell Dirac line, analogous to (5.3). The alternative expressions

$$(\not{p}'+m)\not{\epsilon}(\not{p}+m) = 2(\not{p}'+m)(p'\cdot\varepsilon + \frac{1}{4}\,[\not{\epsilon},\not{q}]) - (p'^2 - m^2)\not{\epsilon} \quad . \tag{5.20a}$$

$$= 2(p\cdot\varepsilon + \frac{1}{4}\,[\not{\epsilon},\not{q}])(\not{p}+m) - \not{\epsilon}\,(p^2 - m^2) \quad , \tag{5.20b}$$

and the decomposition used in (5.3) lead to

$$i\,\frac{1}{\not{p}'-m}\,Q\not{\epsilon}\,\frac{1}{\not{p}-m} = \frac{i}{\not{p}'-m}\,\frac{Q}{p'\cdot q}\,(p'\cdot\varepsilon + \frac{1}{4}\,[\not{\epsilon},\not{q}]) + (-p\cdot\varepsilon - \frac{1}{4}\,[\not{\epsilon},\not{q}])\,\frac{Q}{p\cdot q}\,\frac{i}{\not{p}-m} \quad . \tag{5.21}$$

The Dirac radiation decomposition identity, (5.21), like its scalar counterpart, follows the schematic of Fig. 5 and offers an immediate demonstration of the associated Ward-Takahashi identity.

Parallel to the scalar case, (5.21) provides the correct incoming and outgoing convection and (now) spin currents, for each incoming and outgoing internal line of a given vertex in a source graph, in order that the null zone cancellations go through. The use of radiation decomposition identities yields the general radiation vertex expansion [cf. (2.5) and (5.5)], the sum over source vertices v,

$$M_\gamma(T_G) = \sum_v M_\gamma\,[V_G(v)]\,R(v) \qquad , \tag{5.22}$$

where $M_\gamma[V_G(v)]$ is the radiation vertex amplitude now including internal legs. For the internal legs of a given vertex v, we replace the corresponding spinors (w or w') in (5.10) by spin indices that are tied to the remaining factor, $R(v)$, which contains all propagators. $R(v)$ is T_G less the vertex v, with momentum assignments consistent with photon emission (q) from v.

Since we have seen in (5.7) that internal and external $Q/p\cdot q$ factors are equal in the null zone and since we have a complete set of convection and Dirac spin currents, the conservation of momentum (modulo q) and the rank-zero nature[19,20] of the string of Γ_i's at each vertex v lead to an off-shell version of (5.19). The theorem is thus proven for scalar-spinor tree source graphs with constant couplings.

Since any deviation from minimal coupling for Dirac particles ruins the $n=3$ factorization,[5] it is expected to undermine the radiation interference theorem. In detail, we see that an anomalous magnetic moment coupling[21] (Pauli moment) leads to the modified photon-spinor vertex,

$$\not{\epsilon} \to \not{\epsilon} + \frac{a}{4m}\,[\not{\epsilon},\not{q}] \qquad , \tag{5.23}$$

where the magnetic moment and gyromagnetic ratio are

$$\mu = g\,\frac{e}{2m}\ ,\ g = 2(1+a) \qquad , \tag{5.24}$$

in terms of the anomaly $\underline{a}$. The external current, (5.9a), for example, is then changed to

$$\frac{Q}{p\cdot q}\,\bar{u}(p)\left(p\cdot\varepsilon + \frac{1}{4}\,[\not\varepsilon,\not q](1+a) + \frac{a}{2m}\,\omega_{\mu\nu}\,p^{\mu}\gamma^{\nu}\right) \qquad . \tag{5.25}$$

The previous argument, where $a = 0$, depended on the relationship between the spin currents and a universal Lorentz transformation. The p dependence of the new term , $\omega_{\mu\nu}\,p^{\mu}\gamma^{\nu}$, destroys this relationship.

Therefore, we observe that the Dirac electromagnetic coupling (minimal coupling), as given by the local gauge algorithm that generates a renormalizable theory and such that $a = 0$ in lowest order, is required for the radiation interference theorem. (This should not be confused with the fact that, even without derivatives, the source graph may derive from nonrenormalizable interactions, such as 5-particle couplings.) It is perhaps more to the point, particularly in view of the next subsection, to simply say that $\underline{g = 2}$ is required at the tree graph level for particles with spin. The result that only gauge-theoretic spin currents produce the necessary universal Lorentz transformation is very important and is discussed again in Sec. IX.

This last discussion serves to show that electromagnetic gauge invariance (invariance under $\varepsilon \to \varepsilon + q$) is not sufficient. The Pauli terms, for example, are gauge invariant, but lead to $g \neq 2$, nonrenormalizability, and a violation of the radiation interference theorem, all of which appear to be intimately related to one another.

On the other hand, an important feature of (5.22) is that the terms are separately gauge invariant. This is a consequence of the use of the decomposition identities for internal radiation, in which the Ward-Takahashi identity is made manifest. The example given later displays this feature. The zero-charge limiting case for internal and external Dirac lines can also be studied with that example, and involves interesting subtleties that are discussed generally in Sec. VII.

C. Including spin-one particles

We further extend the proof of the radiation interference theorem to include vector (and axial vector) particles, continuing to use the previous strategy. We add an arbitrary number N of vectors to the 2D Dirac particles and $n-2D-N$ scalars in the tree source graph, but still with no derivative couplings.

The only derivative couplings, therefore, are in the scalar and vector electromagnetic currents; the latter photon coupling is the new ingredient here and has the form[22] of the locally gauge invariant Yang-Mills (gauge theoretic) trilinear vertex, the general expression for which is given Fig. 7. Such a photon-vector-vector coupling corresponds to $\kappa = 1$ for the magnetic moment parameter of the vector particle, or $g = 2$, and is crucial to the validity of the theorem. (Violations for $\kappa \neq 1$ are discussed at the end of this subsection.) The quadrilinear vector couplings of non-Abelian gauge theory, in which the photon participates, can be regarded as seagull terms and are treated in this way in subsection D.

The incorporation of neutral vector particles into the proof is particularly delicate. This problem is correlated with the zero-mass limit and both are considered together in Sec. VII.

The vertex source graph is generalized from (5.8) to

$$V_G(n,D,N) = \prod_{\ell=1}^{N} \eta_\ell^{\mu_\ell} \left(\prod_{i=1}^{D} \bar{w}_i \Gamma_i w_i \right)_{\mu_1 \mu_2 \ldots \mu_N} , \tag{5.26}$$

in terms of the vector polarization factors[13] η. For convenience, we now include possible $g_{\mu\nu}$, $\varepsilon_{\mu\nu\sigma\rho}$ tensors along with the Dirac matrices in the definition of the Γ_i, making up the (constant) rank-N Lorentz tensor into which the η_ℓ are contracted.

As before, the absence of derivative couplings means that only external-line photon attachments contribute to the radiation amplitude generated by (5.26). The vector counterparts of the photon emission factors (5.1) and (5.9) are calculated by contracting the vector propagator, $iP_{\mu\nu}(p)/(p^2 - m^2)$, where

$$P_{\mu\nu}(p) \equiv - g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2} , \tag{5.27}$$

with the photon vertex that is inferred from Fig. 7.

For an external vector leg with charge Q flowing along momentum p and with polarization $\eta(p)$ $(\eta \cdot p = 0)$, the currents (convection plus spin) are

$$\text{outgoing particle:} \quad \frac{Q}{p \cdot q} (p \cdot \varepsilon\, \eta_\mu + \omega_{\mu\nu} \eta^\nu) , \tag{5.28a}$$

$$\text{incoming particle:} \quad (-p \cdot \varepsilon\, \eta_\mu + \omega_{\mu\nu} \eta^\nu) \frac{Q}{p \cdot q} . \tag{5.28b}$$

These replace η_μ in the original source graph. The remarkable simplicity of (5.28), directly related to the lack of explicit mass dependence, is spoiled by the nongauge coupling discussed at the end of this section.

We learn from (5.28) that the relationship between spin currents and the universal Lorentz transformation is not just an accidental aspect of Dirac particles, since the vector spin currents are also proportional (for $g=2$) to the first-order change under (5.12) in their associated wave functions. This relationship is the key to the null zone cancellation, analyzed below, of radiation amplitudes including spin-one particles, and its general features are the subject of Sec. IX.

From (5.1), (5.9), and (5.28) it follows that (5.10) generalizes to

$$\begin{aligned} M_\gamma[V_G(n,D,N)] &= V_G(n,D,N)\ A_{IR}(k,n) \\ &+ \prod_{\ell=1}^{N} \eta_\ell^{\mu_\ell} \Big(\sum_{i=1}^{D} S_i \prod_{j\neq i}^{D} \bar{w}_j' \Gamma_j w_j \Big)_{\mu_1 \cdots \mu_N} \\ &+ \sum_{\ell=1}^{N} \frac{Q_\ell}{p_\ell \cdot q}\, \omega^{\mu_\ell}{}_{\nu} \eta_\ell^{\nu} \prod_{r\neq \ell} \eta_r^{\mu_r} \Big(\prod_{i=1}^{D} \bar{w}_i' \Gamma_i w_i \Big)_{\mu_1 \cdots \mu_N} \quad , \end{aligned} \tag{5.29}$$

with the familiar classical amplitude (3.13) as the repository of the complete set of convection currents, one for each leg, which vanishes under (4.1). In the null zone, the Dirac spin currents in the second term of (5.29) are proportional to the first-order universal transformation of the rank-N spinor product according to the remarks in subsection B and are therefore cancelled by the third term which is similarly related to the first-order transformation of the rank-N vector polarization product. The total first-order change of the rank-zero V_G vanishes under (4.1) by its Lorentz invariance:

$$M_\gamma [V_G] = 0 \qquad \text{(null zone)} \quad . \tag{5.30}$$

We need a radiation decomposition identity for a real photon attached to an off-shell vector line analogous to (5.3) and (5.21) in order to consider an arbitrary tree source graph T_G. This identity is

$$\frac{-iQI_{\gamma\delta}}{(p'^2-m^2)(p^2-m^2)}$$

$$= \frac{iP_{\gamma\beta}(p')}{p'^2-m^2}\,\frac{Q}{p'\cdot q}\,(p'\cdot\varepsilon\, g^{\beta}_{\delta}+\omega_{\delta}{}^{\beta})+(-p\cdot\varepsilon g^{\alpha}_{\gamma}+\omega_{\gamma}{}^{\alpha})\frac{Q}{p\cdot q}\,\frac{iP_{\alpha\delta}(p)}{p^2-m^2} \quad , \qquad (5.31)$$

where, using the notation of Fig. 7,

$$I_{\gamma\delta} \equiv P_{\gamma\beta}(p')\; Y^{\beta\sigma\alpha}(p',q,-p)\;\; P_{\alpha\delta}(p)\,\varepsilon_{\sigma} \quad . \qquad (5.32)$$

Eq. (5.31) is derived using both of the alternate expressions for (5.32)

$$I_{\gamma\delta} = -2P_{\gamma\beta}(p')(p'\cdot\varepsilon\;\, g^{\beta}_{\delta}+\omega_{\delta}{}^{\beta})+\frac{1}{m^2}\,(p'^2-m^2)(\varepsilon_{\gamma}\,p_{\delta}+p'_{\gamma}\,\varepsilon_{\delta}) \quad , \qquad (5.33a)$$

$$= 2(-p\cdot\varepsilon\;\, g^{\alpha}_{\gamma}+\omega_{\gamma}{}^{\alpha})\,P_{\alpha\delta}(p)+\frac{1}{m^2}\,(\varepsilon_{\gamma}\;\,p_{\delta}+p'_{\gamma}\,\varepsilon_{\delta})\;(p^2-m^2) \quad . \qquad (5.33b)$$

and is also described by Fig. 5.

The decomposition (5.31) allows us to form a radiation vertex expansion [cf. (5.22)] in the same manner as before but which now includes internal and external vector particles. For every internal particle with spin that is attached to a given vertex v of T_G, the factor $V_G(v)$, defined as in (5.26), has a free index in place of the spinor or polarization vector, leading to an overall tensor-matrix rank for V_G. The off-shell radiation amplitude $M_{\gamma}[V_G(v)]$, defined in subsection B, is likewise multi-spinor-indexed and a Lorentz tensor.

Now, we may regard V_G (and M_{γ}) as Lorentz invariants in a manner following the spinor description.[20] For each internal vector leg, index μ, we rewrite $(V_G)_{\mu}$ as $(V_G)_{\delta}\eta^{\delta}(\mu)$ for $\eta^{\delta}(\mu) = g^{\delta}_{\mu}$, defining an internal vector wave function. If all wave functions, vector and spinor, external and internal, are universally Lorentz transformed, V_G is unchanged and, in particular, the first-order terms cancel. Since (5.31) provides exactly these internal first-order changes, M_γ continues to satisfy (5.19).

In sum, the spin currents associated with each tree vertex cancel out in the null zone by Lorentz invariance, independently of the convection cancellation. The theorem is thereby extended to scalar-spinor-vector tree source graphs with constant couplings. The detailed example in subsection E includes both internal and external vector particles.

A non-gauge-theoretic photon coupling to vector particles can be seen to spoil the cancellation. For $\kappa \neq 1$, the previous gauge-symmetric vertex for $(p,\alpha)\to(p',\beta)+(q,\mu)$ is augmented by the term[22]

$$i\,Q(\kappa-1)(g_{\beta\mu}q_\alpha - q_\beta\, g_{\mu\alpha}) \quad . \tag{5.34}$$

The currents are changed by the addition of

$$\frac{Q}{p\cdot q}\,\frac{\kappa-1}{2}\,P_{\mu\nu}(\ell)\omega^{\nu\rho}\eta_\rho \quad , \tag{5.35}$$

where $\ell = p+q\ (p-q)$ for the first (second) factor in (5.28). The p dependence of $P_{\mu\nu}$ in (5.35) ruins the universality of the spin currents. We need $\kappa = 1$, or $g = 2$, in the vector magnetic moment,

$$\mu = g\,\frac{e}{2m}\ ,\ g = 1+\kappa \quad , \tag{5.36}$$

in order to maintain the relationship between the spin currents and the universal Lorentz transformation (5.12) and thereby the validity of the radiation interference theorem.

D. Including derivative couplings : seagulls

Although derivatives have already been introduced in the scalar and vector electromagnetic currents, it remains to consider the possibility of derivative couplings in the interactions of the source particles themselves. Such couplings appear naturally in gauge theories, and have been described in Sec. II. In this subsection we show that the radiation theorem continues to hold for the general class of gauge-theoretic interactions in the source graph, and that the current associated with the presence of a derivative coupling is described by the same Lorentz transformation that characterizes spin currents.

We first examine the consequences of single-derivative factors: Lagrangian interactions of the form $(\partial_\mu \Psi_i)(\Psi_j \Psi_k \ldots)^\mu$ or <u>products</u> thereof, $(\partial_\mu \Psi_i)(\partial_\nu \Psi_j)\ldots$, where each field Ψ , boson or fermion, has at most one derivative. Obviously, these include interactions that can be brought into single-derivative form through an integration-by-parts.

Electromagnetic gauge invariance is maintained by the replacement, $\partial_\mu \to \partial_\mu - i Q_i A_\mu$, in terms of the photon field A_μ and the charge Q_i of the field Ψ_i , resulting in the familiar seagull interactions involving the photon. Therefore, in the construction of radiation amplitudes, any momentum-dependent source vertex requires a direct photon attachment, adding a seagull current to the convective and spin currents.

Let us consider a vertex in which there is a derivative coupling, $(\partial^\beta \Psi)\ldots$, and the external <u>or</u> internal leg (particle of Ψ) connected to this vertex, as an isolated part of a source tree graph. In momentum space, the vertex may be denoted by $p^\beta r_\beta$, in terms of the momentum p of the leg and the remaining vertex factors r.

We focus on the radiation, due to the particle Ψ, from this isolated vertex-leg system, as in Fig. 8. This contribution to the vertex radiation amplitude in (5.22) is

$$M_\Psi = [\pm \frac{Q}{p\cdot q}\, p\cdot\varepsilon(p\pm q)^\beta - Q\,\varepsilon^\beta + \text{spin term}]r_\beta \tag{5.37}$$

for an outgoing (+)/incoming(-) particle. In the internal-leg case recall that only the radiation-decomposition term relevant to this vertex is to be included. Aside from a possible external wave function, r resembles R in (5.22) in that it can be expressed entirely in terms of momenta other than p and q.

The seagull term in (5.37) comes from the vertex factor, $-Qg^{\mu\beta}$, which can be directly derived by the constraint of gauge invariance. In this regard, we note that the spin currents are separately gauge invariant and that the convective current in (5.37), $\pm Q\, p\cdot\varepsilon\, p\cdot r/p\cdot q$, is conjointly gauge invariant with the other convection currents in the radiative vertex amplitude.

The <u>seagull</u> and <u>momentum-shift</u> contributions to (5.37) can be rewritten in the suggestive form

$$\frac{Q}{p\cdot q}\,[p\cdot\varepsilon\, q^\beta - p\cdot q\,\varepsilon^\beta]\, r_\beta \quad . \tag{5.38}$$

These terms go hand-in-hand for any single-derivative coupling in the source graph. (They also appear together in first-order q for higher derivatives.)

The significance of (5.38) is that it allows us to identify a <u>universal contact current</u>,

$$\frac{Q}{p\cdot q}\,\omega^{\mu\nu} \quad , \tag{5.39}$$

for photon emission from a line coupled through a (linear) derivative coupling to a vertex, to be added to the convection and (any) spin currents. The rule[23] is that (5.39) replaces $g^{\mu\nu}$ in the derivative coupling, $p^{\mu} = g^{\mu\nu} p_{\nu}$. (We recall that the other currents may be regarded as wave-function substitutions.) A summary of all photon emission factors is given in Table I.

We see from (5.39) that the contact current is proportional to the first-order Lorentz transformation (5.12) of the rank-one derivative. We have previously found that the spin currents transform the wave functions and now we see that the contact currents transform the derivatives. The arguments of the preceding subsections continue to apply: Lorentz invariance guarantees a cancellation of the terms that are first-order in q. [Inasmuch as $\omega^{\mu\nu}$ is linear in q, the order of q is equivalent to the order of λ in (5.12).] In the null zone, the radiation vertices in (5.22) vanish up to $O(q^2)$, in the coefficient of $Q/p\cdot q$.

The $O(q^2)$ terms arise when a spinning particle encounters its own derivative coupling,[24] specifically from the product of the spin current and the momentum shift:

$$\text{spin term} = \text{spin current} \times (p \pm q)^{\beta} \quad , \tag{5.40}$$

from (5.37). Lorentz invariance guarantees only that the first-order term in (5.40) is cancelled in the null zone. Thus, second-order terms develop for interactions in which there are derivatives of Dirac or vector fields, as well as those in which higher derivatives of scalar fields occur. These second-order terms do not cancel in the null zone (and therefore the radiation interference theorem does not hold) in these cases unless an additional mechanism is operative.

In fact, there is an exceptional case in which such an additional mechanism is present. The quadratic terms cancel under (4.1) for the trilinear single-derivative vector-boson vertex of Fig. 7, as a consequence of both the cyclic symmetry of the vertex and of the specific form (5.12) of the universal transformation, $\omega^{\mu\nu}$. This cancellation is demonstrated explicitly in the next subsection and appears to be intimately related to the question of renormalizability, although the theorem is likewise true for a class of nonrenormalizable interactions. That is, the arguments, seen in the past few sections, also go through for couplings involving factored <u>products</u> of single derivatives of distinct scalar fields and of the triplet Yang-Mills structure as well as of any number of scalar, Dirac, or vector fields with constant couplings. Except for the special considerations in Sec. VII involving neutral particles and the proof of the $O(q^2)$ cancellation for the trilinear vector vertex given below, this completes the proof of the radiation interference theorem for the class of gauge theoretic interactions.

E. Example with Yang-Mills vertex

Here we present a detailed example of the radiation interference theorem. We consider the $n=9$ source graph, T_G(Fig. 9), shown in Fig. 9, for nine reasons. The structure is designed to demonstrate the properties of the cyclic trilinear Yang-Mills vertex and its seagull, the latter usually given as a quadrilinear in the gauge theory rules. It also illustrates a tensor $\sigma_{\mu\nu}$ Dirac current, multi-field vertices, a product of two scalar single-derivative couplings, their seagulls, a Levi-Civita tensor $\varepsilon_{\mu\nu\sigma\rho}$ vertex, and vanishing charges for internal vector and Dirac particles. (The zero-charge limit of this example will be discussed in Sec. VII.) In addition, the radiation-vertex expansion and its notation are utilized, and its gauge invariance is displayed.

The amplitude for the source graph can be written in terms of the notation of Fig. 7 as

$$T_G(\text{Fig.}9) =$$

$$= \eta_2^\alpha Y_{\alpha\beta\gamma}(p_2,p_3,-p_1)\eta_1^\gamma V^{\beta\gamma}(p_3)\eta_4^\zeta \bar{u}_5 \sigma_{\gamma\zeta} \frac{1}{\not{p}_6-m_6} v_7 \varepsilon_{\mu\nu\sigma\rho} \eta_8^\mu \eta_9^\nu p_{10}^\sigma p_{11}^\sigma \quad , \qquad (5.41)$$

where $\eta_i \equiv \eta(p_i)$, $\bar{u}_5 \equiv \bar{u}(p_5)$, $v_7 \equiv v(p_7)$, and the vector propagator is $V^{\beta\lambda}(p_3) \equiv (g^{\beta\lambda} - p_3^\beta p_3^\lambda/m_3^2)/(p_3^2-m_3^2)$. (Overall constants are disregarded throughout this section.) Before photon emission, the momenta are related by

$$p_1 - p_2 = p_3 = p_4 + p_5 - p_6 \quad ,$$
$$p_6 = p_{10} + p_{11} - p_7 - p_8 - p_9 \quad . \qquad (5.42$$

Charge conservation leads to the same equations, but with $p_i \to Q_i$.

The radiation amplitude corresponding to (5.41) has the radiation-vertex expansion

$$M\,(\text{Fig. }9) = \sum_{v=1}^{3} M(v)\,R(v) \quad , \tag{5.43}$$

where the notation in (5.22) has been simplified and where $v = 1,2,3$ refers to the vertex at the top, middle, bottom, respectively, of Fig. 9. The vertex radiation amplitude, M(v), can be obtained using the appropriate current insertions, which have been found in the preceding subsections and which are summarized in Table I. The theorem is verified if each M(v) vanishes in the null zone.

The first vertex radiation amplitude is constructed using the external currents, (5.28), and the first (outgoing) internal current in (5.31) with $p' = p_3$, all augmented by contact currents, (5.39), for the momenta in the Yang-Mills vertex. (The contact currents include the quadrilinear γVVV vertex.) The result is

$$\begin{aligned}
M_\beta(1) = Y_{\alpha\tau\gamma}(p_2,p_3,-p_1)\ &\Big[\frac{Q_1}{p_1\cdot q}\,\eta_2^\alpha\,(-p_1\cdot\varepsilon\eta_1^\gamma + \omega^\gamma{}_\delta\eta_1^\delta)g_\beta^\tau \\
&+ \frac{Q_2}{p_2\cdot q}\,(p_2\cdot\varepsilon\ \eta_2^\alpha + \omega^\alpha{}_\delta\,\eta_2^\delta)\,\eta_1^\gamma g_\beta^\tau + \frac{Q_3}{p_3\cdot q}\,\eta_2^\alpha\eta_1^\gamma\,(p_3\cdot\varepsilon\, g_\beta^\tau + \omega^\tau{}_\beta)\Big] \\
&+ \eta_2^\alpha\,\eta_1^\gamma\,\Big[\frac{Q_1}{p_1\cdot q}\,(g_{\gamma\alpha}\,\omega_{\beta\tau} - g_{\beta\gamma}\,\omega_{\alpha\tau})p_1^\tau \\
&+ \frac{Q_2}{p_2\cdot q}\,(g_{\gamma\alpha}\,\omega_{\beta\tau} - g_{\alpha\beta}\omega_{\gamma\tau})\,p_2^\tau + \frac{Q_3}{p_3\cdot q}\,(g_{\alpha\beta}\omega_{\gamma\tau} - g_{\beta\gamma}\omega_{\alpha\tau})p_3^\tau\Big] \\
&+ \frac{Q_1}{p_1\cdot q}\,\eta_2^\alpha\,\omega^\gamma{}_\delta\eta_1^\delta(g_{\beta\gamma}q_\alpha - g_{\gamma\alpha}q_\beta) + \frac{Q_2}{p_2\cdot q}\,\eta_1^\gamma\omega^\alpha{}_\delta\eta_2^\delta(g_{\gamma\alpha}q_\beta - g_{\alpha\beta}q_\gamma) \\
&\qquad + \frac{Q_3}{p_3\cdot q}\,\eta_2^\alpha\eta_1^\gamma\omega^\tau{}_\beta(g_{\alpha\tau}q_\gamma - g_{\tau\gamma}q_\alpha)
\end{aligned} \tag{5.44}$$

with its common factor in (5.43) given by the remainder of (5.41),

$$R^{\beta}(1) = V^{\beta\lambda}(p_3)\eta_4^{\zeta} \cdots \quad . \tag{5.45}$$

Note the indicial communication between (5.44) and (5.45). Now

$$p_1 - p_2 - q = p_3 \tag{5.46}$$

with the rest of (5.42) unchanged.

$M_{\beta}(1)$ is easily checked to be gauge invariant, since $\omega^{\mu\nu}$ vanishes under the replacement $\varepsilon \to q$ and since charge is conserved. This property holds for the other vertices as well and is an important consequence of the use of the radiation decomposition identities, as we have already noted. (See subsection B.) In this regard, it is crucial that the decomposition (5.31) produces the same outside factor (5.45) as do the external leg attachments. Momentum dependence has to be considered carefully inasmuch as the internal momenta change, depending upon the photon's origin.

In the null zone, we find

$$\begin{aligned} M_{\beta}(1) = \frac{Q_1}{p_1\cdot q} \{ & Y_{\alpha\tau\gamma}(p_2,p_3,-p_1)[\eta_2^{\alpha}\eta_1^{\gamma}(-p_1+p_2+p_3)\cdot\varepsilon\, g_{\beta}^{\tau} \\ & + \eta_2^{\alpha}\omega^{\gamma}{}_{\delta}\eta_1^{\delta}g_{\beta}^{\tau} + \omega^{\alpha}{}_{\delta}\eta_2^{\delta}\eta_1^{\gamma}g_{\beta}^{\tau} + \eta_2^{\alpha}\eta_1^{\gamma}\omega^{\tau}{}_{\beta}] \\ & + \eta_2^{\alpha}\eta_1^{\gamma}\ [g_{\alpha\beta}\omega_{\gamma\tau}(p_3-p_2)^{\tau} + g_{\beta\gamma}\omega_{\alpha\tau}(-p_1-p_3)^{\tau} + g_{\gamma\alpha}\omega_{\beta\tau}(p_2+p_1)^{\tau}] \\ & + 2\omega_{\delta\beta}\eta_2^{\delta}\eta_1\cdot q + 2\omega_{\gamma\delta}\eta_2^{\delta}\eta_1^{\gamma}q_{\beta} + 2\omega_{\beta\delta}\eta_1^{\delta}\eta_2\cdot q\} \qquad \text{(null zone)}\ , \end{aligned} \tag{5.47}$$

grouping the quantities inside the curly bracket according to powers of q.

In fact, $M_{\beta}(1) = 0$ in the null zone. This complete cancellation can be described order-by-order in q.

First, the zeroth-order convection currents obviously cancel. The next six terms, linear in q, are the first-order universal Lorentz rank-one changes in the external vector wave functions, in the internal vector wave function ($\omega^{\tau}{}_{\beta}$ term) defined by $M_{\beta} \equiv M_{\tau}\eta^{\tau}(\beta)$ with $\eta^{\tau}(\beta) = g^{\tau}_{\beta}$, and in the four-momenta of the vertex, respectively. Since these are all contracted together, sometimes through the numerically invariant $g_{\mu\nu}$, Lorentz invariance guarantees their cancellation, and an explicit calculation using the antisymmetry of $\omega_{\mu\nu}$ bears this out.

We call special attention to the cancellation of the last three terms, quadratic in q, in (5.47). This goes beyond Lorentz invariance, requiring the cyclic symmetry of the trilinear vertex and the specific structure of $\omega_{\mu\nu}$ in (5.13).

The second vertex radiation amplitude is constructed using (5.9a),(5.28a), and the second (incoming) terms in (5.31) and (5.21), yielding

$$
\begin{aligned}
M_{\lambda}(2)_{\alpha} = \bar{u}_5 \{ & \frac{Q_4}{p_4 \cdot q} (p_4 \cdot \varepsilon\, \eta_4^{\zeta} + \omega^{\zeta}{}_{\nu}\eta_4^{\nu})\sigma_{\lambda\zeta} \\
& + \frac{Q_3}{p_3 \cdot q} (-p_3 \cdot \varepsilon\, g^{\nu}_{\lambda} + \omega^{\nu}{}_{\lambda})\eta_4^{\zeta}\sigma_{\nu\zeta} + \frac{Q_5}{p_5 \cdot q} (p_5 \cdot \varepsilon + \tfrac{1}{4}[\not{\varepsilon}, \not{q}])\eta_4^{\zeta}\sigma_{\lambda\zeta} \\
& + \frac{Q_6}{p_6 \cdot q} \eta_4^{\zeta}\sigma_{\lambda\zeta}(-p_6 \cdot \varepsilon - \tfrac{1}{4}[\not{\varepsilon}, \not{q}]) \}_{\alpha} \quad ,
\end{aligned}
\tag{5.48}
$$

with the contracted remainder (pre- and post-multipliers)

$$
R^{\lambda}(2)_{\alpha} = \ldots\ V^{\beta\lambda}(p_3)\left(\frac{1}{\not{p}_6 - m_6} v_7\right)_{\alpha} \quad , \tag{5.49}
$$

and with (5.42) modified by

$$p_3 = p_4 + p_5 - p_6 + q \quad . \tag{5.50}$$

The gauge invariance of (5.48) is easily seen.

It is now easy to see that M(2) also vanishes in the null zone. The Dirac spin currents produce the first-order Lorentz transformation of $\sigma_{\lambda\zeta}$ [cf. (5.17)],

$$\Delta\sigma_{\lambda\zeta} \equiv \frac{1}{4}\,[[\not{\varepsilon},\not{q}],\sigma_{\lambda\zeta}] = \frac{1}{2}\,\not{\varepsilon}\,[\not{q},\sigma_{\lambda\zeta}] + \frac{1}{2}\,[\not{\varepsilon},\sigma_{\lambda\zeta}]\not{q}$$

$$= \omega_\lambda{}^\beta \sigma_{\beta\zeta} + \omega_\zeta{}^\beta \sigma_{\lambda\beta} \quad , \tag{5.51}$$

which is cancelled by the (vector wave function) ω terms in (5.48)

Finally, the third vertex radiation amplitude is constructed using (5.1b),(5.9c),(5.28a),(5.39), and the first (outgoing) term in (5.21). The gauge-invariant result is

$$\begin{aligned} M(3)_\beta = \; & \varepsilon_{\mu\nu\sigma\rho}\,\{\eta_8^\mu \eta_9^\nu p_{10}^\sigma p_{11}^\rho\,[\frac{Q_6}{p_6\cdot q}\,(p_6\cdot\varepsilon + \frac{1}{4}[\not{\varepsilon},\not{q}]) \\ & + \frac{Q_7}{p_7\cdot q}(p_7\cdot\varepsilon - \frac{1}{4}\,[\not{\varepsilon},q])] + p_{10}^\sigma p_{11}^\rho\;[\frac{Q_8}{p_8\cdot q}(p_8\cdot\varepsilon\,\eta_8^\mu + \omega^\mu{}_\alpha \eta_8^\alpha)\eta_9^\nu \\ & + \frac{Q_9}{p_9\cdot q}(p_9\cdot\varepsilon\eta_9^\nu + \omega^\nu{}_\alpha\eta_9^\alpha)\eta_8^\mu] + \eta_8^\mu\eta_9^\nu\;[\frac{Q_{10}}{p_{10}\cdot q}\,(-\,p_{10}\cdot\varepsilon\, p_{10}^\sigma + \omega^\sigma{}_\alpha p_{10}^\alpha)\,p_{11}^\rho \\ & + \frac{Q_{11}}{p_{11}\cdot q}\,(-\,p_{11}\cdot\varepsilon\, p_{11}^\rho + \omega^\rho{}_\alpha p_{11}^\alpha)\,p_{10}^\sigma]\,\}_\beta v_7 \quad , \end{aligned} \tag{5.52}$$

with the contracted spin row-matrix factor

$$R(3)_\beta = (\cdots \frac{1}{\not{p}_6 - m_6})_\beta \quad . \tag{5.53}$$

Now (5.42) is modified by

$$p_6 = p_{10} + p_{11} - p_7 - p_8 - p_9 - q \quad . \tag{5.54}$$

M(3) is also seen to vanish under (4.1) noting first the direct cancellation of the Dirac spin currents as expected for a scalar fermion coupling. The cancellation of the remaining contact and vector spin currents, expected by the Lorentz invariance of the remaining coupling, follows from the use of the basic identity

$$g^{\mu\nu}\varepsilon^{\alpha\beta\gamma\sigma} = g^{\mu\alpha}\varepsilon^{\nu\beta\gamma\sigma} + g^{\mu\beta}\varepsilon^{\alpha\nu\gamma\sigma} + g^{\mu\gamma}\varepsilon^{\alpha\beta\nu\sigma} + g^{\mu\sigma}\varepsilon^{\alpha\beta\gamma\nu} \quad . \tag{5.55}$$

For example,

$$\omega^{\mu}{}_{\alpha}\eta_8^{\alpha}\varepsilon(\mu,9,10,11) + \omega^{\nu}{}_{\alpha}\eta_9^{\alpha}\varepsilon(8,\nu,10,11) + \omega^{\sigma}{}_{\alpha}p_{10}^{\alpha}\varepsilon(8,9,\sigma,11)$$

$$+ \omega^{\rho}{}_{\alpha}p_{11}^{\alpha}\varepsilon(8,9,10,\rho) = 0 \quad , \tag{5.56}$$

where $\varepsilon(\mu,9,10,11) \equiv \varepsilon_{\mu\nu\sigma\rho}\eta_9^{\nu}p_{10}^{\sigma}p_{11}^{\rho}$, etc. We conclude that the full amplitude in (5.43) vanishes, M_{γ}(Fig. 9) = 0, in the null zone.

We leave as a simple exercise for the reader the demonstration of how the theorem is violated, by, say, an $n = 3$, double-derivative coupling $V^{\alpha}V^{\beta}(\partial_{\alpha}\partial_{\beta}S)$ source vertex with scalar (vector) fields $S(V)$. In contrast to $M_{\gamma}(1)$, the terms in quadratic in q do not cancel in this example; note that an integration-by-parts can rearrange in the interaction into single-derivative form, but then we would have a derivative of a vector field and the second-order terms still arise. No symmetry leads to the cancellation of these $O(q^2)$ terms.

VI. RADIATION REPRESENTATION

If M_γ is a radiation amplitude generated by a tree source graph, it is necessarily linear in the charges of the external particles (over and above the original source graph couplings). Hence, if M_γ also satisfies the radiation interference theorem, it has an $(n-2)$-dimensional first-order zero in the space of the $Q/p\cdot q$ factors. [There is no ambiguity in the order of the zero or in any analytic continuation, since the radiation vertex expansion (5.22) is explicitly linear in the $Q/p\cdot q$ factors.] In this section, we establish a new representation of such radiation amplitudes that makes the zero structure manifest.

The conclusions of Sec. V are summarized by the statement that each gauge-theoretic vertex radiation amplitude in (5.22) can be written as

$$M_\gamma(V_G) = \sum_{i=1}^{n_v} \frac{Q_i J_i}{p_i\cdot q} \quad , \tag{6.1}$$

where

$$\sum_{i=1}^{n_v} \delta_i Q_i = 0 \quad , \tag{6.2a}$$

$$\sum_{i=1}^{n_v} J_i = 0 \quad , \tag{6.2b}$$

$$\sum_{i=1}^{n_v} \delta_i p_i\cdot q = 0 \quad . \tag{6.2c}$$

The source vertex subgraph V_G has n_v internal and external legs, whose propagator factors are not included in (6.1). All legs are external in the special case of a vertex source graph $(n_v = n)$.

J_i is the product of the photon-emission current j_i for the i^{th} leg (the j_i rules are summarized in Table I) and the remaining factors of the original vertex amplitude. Examples for J_i appear in (5.44), (5.48), and (5.52). The current sum rule, (6.2b), a consequence of translational, Lorentz, and Yang-Mills symmetries as we recall, is <u>independent of whether or not the null zone condition is realized</u>. In addition, the conservation laws, (6.2a) and (6.2c), are independent of each other and of the currents.

The expression in (6.1) obviously vanishes for identical $Q_i/p_i \cdot q$ by (6.2b), and, alternatively, for identical $J_i/p_i \cdot q$ by (6.2a), which are the circumstances previewed in Sec. II. We also recall that (6.2c), in conjunction with (6.2a) or (6.2b), is responsible for the reduction in the number of independent $Q/p\cdot q$ or $J/p\cdot q$ factors, respectively, via (4.2). We wish to use the algebra underlying these results to find a form for the amplitudes that displays the first-order zeros explicitly, and which is specifically a bilinear expansion in differences of the $Q/p\cdot q$ and $J/p\cdot q$ factors.

The following trivial lemma will help to introduce the algebra:

<u>Lemma 1</u> : If $s = \sum_i a_i b_i$, where $\sum_i b_i = 0$, then $s = \sum_i (a_i - a_j) b_i$, for all j. (The sum may omit $i = j$.)

Proof : Obvious. The zero for identical a_i is now explicit and a similar zero for identical b_i arises for $\sum_i a_i = 0$.[25]

Actually, we need a lemma addressing the specific form of (6.1):

<u>Lemma 2</u> : If[26]

$$\sum_{i=1}^{\ell} A_i = \sum_{i=1}^{\ell} B_i = \sum_{i=1}^{\ell} C_i = 0 \ , \tag{6.3}$$

then

$$\sum_{i=1}^{\ell} \frac{A_i B_i}{C_i} = \sum_{i=1}^{\ell} \left(\frac{A_i}{C_i} - \frac{A_j}{C_j}\right) C_i \left(\frac{B_i}{C_i} - \frac{B_k}{C_k}\right) \quad . \tag{6.4}$$

for all j,k . (The sum may omit $i = j,k$.)

Proof : Multiply out the factors. Writing $A_iB_i/C_i = C_i(A_i/C_i)(B_i/C_i)$, we see that (6.4) simply exhibits the invariance under $A_i/C_i \to A_i/C_i + \text{constant}$, etc.

The expected reduction to only $\ell-2$ differences of A_i/C_i (or B_i/C_i) is most directly effected by choosing $j \neq k$ in (6.4). In the simplest case, $\ell = 3$, we may choose $j = 2$ and $k = 3$ obtaining

$$\sum_{i=1}^{3} \frac{A_iB_i}{C_i} = \left(\frac{A_1}{C_1} - \frac{A_2}{C_2}\right) C_1 \left(\frac{B_1}{C_1} - \frac{B_3}{C_3}\right) \tag{6.5a}$$

$$= \frac{C_1C_2}{C_3}\left(\frac{A_1}{C_1} - \frac{A_2}{C_2}\right)\left(\frac{B_2}{C_2} - \frac{B_1}{C_1}\right) \quad , \tag{6.5b}$$

Any permutation of 123 is permitted in (6.5). Eq. (6.3) has been used in passing from (6.5a) to (6.5b), the factorization formula[27] of Ref. 5.

The application of (6.4) to (6.1) yields the <u>radiation representation</u> of $M_\gamma(V_G)$,

$$M_\gamma(V_G) = \sum_{i=1}^{n_V} \delta_i p_i \cdot q \, \Delta_{ij}(Q) \, \Delta_{ik}(\delta J) \quad , \tag{6.6}$$

where we define the (naturally occurring) differences,

$$\Delta_{ij}(X) \equiv \frac{X_i}{p_i \cdot q} - \frac{X_j}{p_j \cdot q} \quad . \tag{6.7}$$

The freedom in the choice of j,k can again be used, as in (6.5a), to reduce (6.6) to the $n_V - 2$ independent differences[28] among the $\Delta_{ij}(Q)$ and among the $\Delta_{ij}(\delta J)$. From (5.22) and (6.6) we have, therefore, a <u>radiation representation for the general gauge-theoretic radiation amplitude</u>.

The radiation interference theorem and its complement, introduced in Sec. II, are made manifest in the combination of (6.6) and (5.22). This is because the differences (6.7) vanish for each vertex (including internal legs),

$$\Delta_{ij}(Q) = 0 \quad , \tag{6.8}$$

in the null zone. [Recall (5.7).] Similarly,

$$\Delta_{ij}(\delta J) = 0 \quad , \tag{6.9}$$

for identical external $J/p\cdot q$ factors.

Although (6.9) is satisfied in the physical region only under trivial circumstances (for example, $p_i\cdot\varepsilon/p_1\cdot q - p_2\cdot\varepsilon/p_2\cdot q$ vanishes only if $\vec{p}_i \perp \vec{\varepsilon}$ in two-body c.m. scalar scattering), $M_\gamma(V_G)$ can always be expressed in the bi-difference form (6.6) which embodies the consequences of the symmetry properties of the radiation amplitudes. From this perspective, both versions of the radiation theorem are by-products of the radiation representation.

We note that a radiation representation in which only differences in external $Q/p\cdot q$ factors appear can be written for the complete radiation amplitude $M_\gamma(T_G)$. Eq. (4.2) and the linearity in the $Q/p\cdot q$ factors imply that

$$M_\gamma(T_G) = \sum_{i=1}^{n} \frac{Q_i}{p_i\cdot q} I_i(T_G) \quad , \tag{6.10}$$

where I_i is independent of the charges. It follows from the radiation theorem that

$$\sum_i^n I_i = 0 \quad , \tag{6.11}$$

so that Lemma 2 applies. Thus,

$$M_\gamma(T_G) = \sum_{i=1}^{n} S_i p_i\cdot q\, \Delta_{ij}(Q)\, \Delta_{ik}(\delta I) \quad . \tag{6.12}$$

However, the I_i are less convenient for calculation or for physical interpretation. For example, there is no obvious gauge-invariant grouping of terms (6.10) or (6.12). Therefore, we prefer to use (6.6) in combination with (5.22), in which there are the same number of terms, $n-2$, as in (6.12).

That there are the same number of terms follows from the fact that each M_γ in (6.6) can be reduced to $n_v - 2$ terms, through the freedom in j and k, and the fact that, for any tree graph with V total vertices and n external particles,

$$\sum_{v=1}^{V} (n_v - 2) = n - 2 \quad . \tag{6.13}$$

Thus the preferred radiation representation (of the radiation vertex expansion) is in one-to-one correspondence with the minimal $n-2$ terms that are anticipated by the theorem and that are seen explicitly in (6.12).

The economy of the organization of the radiation amplitude into only $n-2$ terms can be appreciated when we realize that there are as many as $2n-3$ radiative graphs arising from external and internal line photon attachments onto a given T_G (with n external lines) and as many as $3(n-2)$ more seagull terms (the maximum of $\sum n_v$). Thus, each helicity amplitude can be simplified, with the symmetries manifest, by the use of the radiation representation, (5.22) with (6.6), particularly in view of the fact that (5.22) is a gauge-invariant decomposition.

Let us illustrate the radiation representation using the example depicted in Fig. 6 and given in Eq. (5.4). Implementing (5.22) and (6.6), we find

$$M_\gamma(\text{Fig. } 6) = \sum_{1}^{2} M(v)\ R(v) \quad , \tag{6.14}$$

where, by choice,

$$M(1) = -p_1\cdot q\left(\frac{Q_1}{p_1\cdot q} - \frac{Q_4}{p_4\cdot q}\right)\left(\frac{p_1\cdot\varepsilon}{p_1\cdot q} - \frac{(p_1-p_4)\cdot\varepsilon}{(p_1-p_4)\cdot q}\right) \quad , \tag{6.15a}$$

$$= -\frac{p_1\cdot q\ p_4\cdot q}{(p_1-p_4)\cdot q}\left(\frac{Q_1}{p_1\cdot q} - \frac{Q_4}{p_4\cdot q}\right)\left(\frac{p_4\cdot\varepsilon}{p_4\cdot q} - \frac{p_1\cdot\varepsilon}{p_1\cdot q}\right) \quad , \tag{6.15b}$$

$$R(1) = \frac{i\lambda_3^2}{(p_3 - p_2)^2 - m_5^2} \quad , \tag{6.16}$$

and $M(2), R(2)$ are obtained by relabeling the charges and momenta in (6.15)-(6.16) according to $1 \to 2, 4 \to 3$. The steps from (6.15a) to (6.15b) follow those in (6.5).

This example will be of use in the next section where external and internal neutral particles are considered. In general, the zero-charge limit is found to be clarified by the use of the radiation representation.

VII. NEUTRAL PARTICLES

In this section we investigate the conditions under which neutral[29] external particles can be included in the radiation interference theorem. We also show that, although there are no new restrictions for neutral internal particles the null zone cancellation goes through differently in such circumstances. The examples of Sec. V and certain reactions in QED illustrate our conclusions.

A. A view of the problem from the radiation representation

The radiation representation of Sec. VI makes it clear that zeros are present in gauge-theoretic radiation amplitudes in tree approximation, even for opposite-sign charges. For example, radiation zeros occur in the lowest-order amplitude for the reaction $e^+e^- \to e^+e^- \gamma$, albeit in the unphysical region. (This reaction is discussed further in Sec. XI.) Charge and momentum conservation, the mass-shell constraints, and Lorentz invariance, which are ingredients of the radiation interference theorem, can be maintained even for the unphysical energies that the null zone condition (4.1) may require.

A cursory look at the radiation representation might lead one to conclude, however, that there would be no radiation zero in the presence of an external particle r with <u>zero charge</u>, $Q_r = 0$. For a set $\{r\}$ of zero external charges in a vertex source graph, the radiation representation (6.6) reduces to

$$M_\gamma(V_G) = \sum_{i \neq r} \delta_i p_i \cdot q \; \Delta_{ij}(Q) \; \Delta_{ik}(\delta J) - \frac{Q_j}{p_j \cdot q} \sum_r \delta_r p_r \cdot q \; \Delta_{rk}(\delta J) \quad , \tag{7.1}$$

where $j,k \neq r$. (Our conclusions are independent of the specific choices of j and k.) The null zone condition, $\Delta_{ij}(Q) = 0$, for the nonzero charges does not imply that $M_\gamma(V_G) = 0$ since only the first term in (7.1) is eliminated. The discrepancy ultimately derives from the fact that the terms $Q_r J_r / p_r q$ are now missing from the amplitude in (6.1).

The cursory conclusion is wrong. To see this note that (7.1) will vanish if in addition to $\Delta_{ij}(Q) = 0$ (for $i,j \neq r$) we have

$$\sum_r p_r\, q = 0 \quad \text{(null zone)} \quad , \tag{7.2}$$

and

$$\sum_r J_r = 0 \quad \text{(null zone)} \quad , \tag{7.3}$$

since then the second term in (7.1) is also eliminated. The first requirement is met if (but not only if - see subsection C) each neutral particle is massless and travels parallel to the photon, precisely the conditions expected from the zero-charge limit,[30] $Q_r \to 0$, of the null zone equations, where simultaneously we must have $p_r \cdot q \to 0$. This discussed in more detail in subsection B where it is shown that, in addition, each J_r vanishes separately when $p_r \propto q$ ($p_r \cdot q = 0$) for a (massless) scalar or Dirac neutral external particle. A massless vector neutral particle can be included only if it is coupled to a conserved current. The second requirement (7.3) is then also satisfied under these conditions.

Therefore, the radiation interference theorem is unaltered by the presence of the prescribed neutral external particles. (We will see in subsection D that neutral internal lines present no problems.) The null zone is simply the corresponding limit of (4.1). The radiation zero in the radiation representation is no longer manifested by Δ_{ij} factors alone, but is also associated with the vanishing or, in the case of neutral internal particles, the cancellation of currents in the radiation representation.

B. One external neutral particle

Suppose that only one external particle r has zero charge, $Q_r = 0$, the rest of the particles having nonzero charges of the same sign. (Particle r may be in the initial or final state.) If the other $n-1$ external particles have equal $Q/p\cdot q$ factors, it necessarily follows that

$$p_r\cdot q = 0 \qquad \text{(null zone)} \quad , \tag{7.4}$$

from (6.2a) and (6.2c), so that

$$p_r = K_r q \tag{7.5}$$

for constant $K_r > 0$. [This is the same condition obtained by the $Q_r \to 0$ limit of the null zone conditions, (4.1).] Therefore, a single external neutral particle of any spin must be massless and must enter or exit the scattering region parallel to the photon, for a physical null zone to exist.[31] This agrees with (4.17) and (A.15) for $Q_1 = 0$, $v_1 = 1$.

In order to have a zero in the radiation amplitude for the vertex to which r is attached, the partial sum must vanish,

$$\sum_{i\neq r} J_i = 0 \qquad \text{(null zone)} \quad , \tag{7.6}$$

since J_r is absent from the sum in (6.1). (We defer the discussion of neutral internal lines to subsection D.) Eqs. (6.2b) and (7.6) imply that

$$J_r = 0 \qquad \text{(null zone)} \quad . \tag{7.7}$$

Hence, even though $Q_r = 0$, its associated J_r is still relevant as a test current for the determination of whether (7.6) is satisfied. It suffices to consider the factors j_r which can be calculated as before with $g = 2$, independently of Q_r, using Table I.

The evaluation of j_r for the different spins leads to the following lemma:

Lemma I : We may include a neutral external particle in the radiation interference theorem provided that it is massless and, in the special case that the particle is a vector boson, it is coupled to a conserved current in a nonforward direction (defined below).

In the proof of the lemma, we first note that the convection current $p_r \cdot \varepsilon$ is zero by (7.5). This is illustrated by letting $Q_1 = 0$ and $p_1 = K_1 q$ in the scalar example of Fig. 6, corresponding to a neutral external scalar particle, whereupon the first radiation vertex amplitude in (5.4) or (6.15) vanishes. (Only one null zone equation needs to be satisfied for a 3-vertex source.)

The Dirac spin current also vanishes by (7.5) since, for example $\not{p}_r u(p_r) = 0$, when $m_r = 0$. Indeed, the neutrino reaction (1.13) has a radiation zero (which is spoiled by an anomalous W magnetic moment)[3] at a location (1.14) given by (4.18). Considering the external Dirac particle 5 in Fig. 9 to be neutral, with $Q_5 = 0$ and $p_5 = K_5 q$, the amplitude (5.48) also can be seen to vanish in the null zone of the remaining particles.

The contact current is likewise zero by (7.5) since it involves the contraction $p_r^\alpha \omega_{\alpha\beta}$. Either particle 10 or 11 in Fig. 9 serves as an example in this case, since they both have derivative couplings. The amplitude (5.52) still vanishes in the null zone for, say, $Q_{10} = 0$ and $p_{10} = K_{10} q$.

Finally, the vector spin current can be written as

$$\omega_{\alpha\beta} \eta_r^\beta = q_\alpha \varepsilon \cdot \eta_r \quad , \tag{7.8}$$

using (7.5). Let us rewrite q_α in (7.8),

$$q_\alpha = (p_r \pm q)_\alpha / (K_r \pm 1) \quad , \tag{7.9}$$

in terms of the momentum transferred to the vertex, which is $p_r \pm q = q\ (K_r \pm 1)$ for photon emission from a particle in the final/initial state. (Replace $q \to -q$, for photoabsorption.) Therefore, (7.8) does not contribute in the event that the vector particle is attached to a conserved current,[32] with $K_r \neq 1$.

This last result implies that we may include additional external photons in the radiation interference theorem. For example, the reaction $e^-e^- \to e^-e^-\gamma\gamma$ has a null zone where the photons are parallel and which is a simple generalization of the null zone for reaction (4.6), $e^-e^- \to e^-e^-\gamma$. The fact that the "first" photon must be coupled to a conserved current requires a gauge-invariant set of source graphs. Also, one could develop a hierarchy of radiation representations by successive application of the procedure in Sec. VI.

A radiation zero is not present in the exceptional case, $K_r = 1$. That is, (7.8) does not vanish and (7.7) does not hold if an initial-state neutral vector particle has momentum identical to the final photon. Such a "forward scattering" transfers no momentum to the vertex and, for an example,[33] let us consider Compton scattering,

$$\gamma + e \to \gamma + e \quad . \tag{7.10}$$

The null zone is the forward direction, where the convection currents cancel, but with zero momentum transfer the spin terms do not. In the forward direction, the amplitude is nonzero and is proportional to

$$e\ \bar{u}(p)\gamma^\mu u(p)\ \frac{e}{p\cdot q}\ \omega_{\mu\nu}\ \varepsilon'^\nu = 2\ e^2\ \varepsilon\cdot\varepsilon' \quad , \tag{7.11}$$

In (7.11), $p_r = q$ and $\eta_r = \varepsilon'$.

Since (6.2b) is based on Poincare invariance (see Sec. IX), we should find a simple picture for (7.7), when it holds, using momentum and angular momentum conservation. The vanishing of the convection current can be attributed to the fact that a scalar particle cannot emit a unit of helicity collinearly. A massless spinor particle cannot flip its helicity with a vector coupling, and neither can a massless vector particle whose longitudinal component has been eliminated. (This component is not eliminated, however, for $K_r = 1$ which is the exceptional case of forward scattering.)

It is also noteworthy that the calculations showing $J_r = 0$ for massless, collinear particles exhibit the same mechanism whereby collinear mass singularities are suppressed in infrared-divergence studies.[34] Related to this is the fact that the $g \neq 2$ photon-emission factors are divergent in the massless limit[35] (see Sec. V). Convergence for $g = 2$ is crucial for the inclusion of neutral particles in the radiation interference theorem.

The question of gauge dependence arises for the evaluation of J_r in the case of a massless neutral vector particle. Since we are after the defect in (6.2b), where it is only the interactions of the nonzero charges that concern us, the question is irrelevant; the unitary-gauge emission factors (5.28) are sufficient for the purpose of evaluating the partial sum (7.6).

Nevertheless, we can show that the emission factors, (5.28), apply in a more general gauge. Working in a general covariant gauge,[36] we replace the propagator factor (5.27) by

$$P_{\mu\nu}(p) = - g_{\mu\nu} + \frac{(1-\xi)p_\mu p_\nu}{p^2 - \xi m^2} \quad , \tag{7.12}$$

with $\xi = 1, 0, \infty$, corresponding to the Feynman, Landau, and unitary ($m \neq 0$) gauges, respectively. The emission factor (5.28a), for example, is replaced by

$$\frac{Q}{p\cdot q}\left[p\cdot\epsilon\eta_\mu + \omega_{\mu\nu}\eta^\nu - \frac{p\cdot q\eta\cdot\epsilon}{2p\cdot q + (1-\xi)m^2}\,(p+q)_\mu\right] \quad , \tag{7.13}$$

where (5.28a) is recovered in the limit $\xi \to \infty$. It is seen that, regardless of the values of ξ and m, the presence of a conserved current eliminates the $(p+q)_\mu$ term in (7.13). The factor (5.28b) is similarly gauge and mass independent, with a conserved current. Thus the vanishing of j_r in the unitary gauge for a massless vector neutral particle coupled to a conserved current holds in any gauge.

C. Additional external neutral particles

The situation is summarized by the following lemma:

Lemma II Lemma I applies independently of the number of neutral external particles.

In proving Lemma II, we simply note that radiation zeros can occur if each neutral particle r satisfies the criteria of Lemma I : Conserved currents for "nonforward" vector particles and masslessness. The zeros arise in the following null zone specialized to a set of neutral particles $\{r\}$:

$$\Delta_{ij}(Q) = 0 \ , \ i,j \neq r \quad , \tag{7.14a}$$

$$p_r\cdot q = 0 \ , \quad (p = K_r q) \ . \tag{7.14b}$$

This null zone is consistent with the limit of (4.1). The set of such neutral particles and the photon can be regarded as a massless composite and can easily be included in the discussion of a physical null zone (see the Appendix). By (7.14b) and the arguments in B, each of the missing currents is zero, so that (7.6) is true for each vertex.

The question we now consider is whether the sufficient conditions (7.14) are also necessary. Could the null zone be larger? To address this suppose that there are $n_o \leq n - 2$ external neutral particles[29] at a given vertex. If the remaining $n - n_0$ particles have the same $Q/p\cdot q$ factor, then the generalization of (7.4) is

$$P\cdot q = 0 \qquad \text{(null zone)} \qquad , \tag{7.15}$$

where P is the total neutral momentum,

$$P \equiv \sum_r^{n_o} \delta_r p_r \quad . \tag{7.16}$$

[Cf. (7.2)] Therefore, P must be light-like, $P \propto q$, if the neutral particles are all in the initial state, or all in the final state. In such cases, each p_r satisfies (7.14b).

We now consider the alternative possibility corresponding to neutral particles in both the initial and final states, where (7.15) does not lead to (7.4) for the individual particles. However, we still require (7.6) for each vertex, so that the sum over the currents J_r for the neutral particles at each vertex must vanish by (6.2b). Postponing the possibility of neutral internal particles until the next subsection, the vanishing, for arbitrary photon polarization, of the total convection current in this sum, $p\cdot\varepsilon$, necessitates $P \propto q$. (It is to be emphasized again that a radiation zero, as we have defined it, refers to cancellations that are not peculiar to the various polarization states.) The spin and contact currents could cancel by Lorentz invariance. The conclusion is that we can augment (7.14) but only by configurations where the momentum transfer is lightlike and where the neutral sector in each vertex factorizes in a Lorentz invariant manner such that its spin and contact currents are not needed to cancel the currents in the charge sector.

D. Internal neutral particles

We now verify that the radiation interference theorem holds without qualification for neutral internal tree lines, as it might be expected in view of the fact that the null zone condition involves only the external particles. The limit $Q_I \to 0$ after the imposition of the null zone condition (4.1) obviously shows the standard cancellation within each vertex, in terms of the radiation vertex expansion, (5.22), where the appropriate radiation decomposition identity (see Table I) has been used, with $Q_I/p_I \cdot q$ nonzero and equal to the external $Q/p \cdot q$ factors.

The case in which we are interested, however, is $Q_I = 0$, ab initio, which will be shown to involve cancellations between vertices. From the absence of the photon coupling to each neutral I, there is a $\tilde{J}_I$ missing in the expression (6.1) for each of the pair of vertex amplitudes. The fact that the two vertices now conspire in the null zone cancellation can be stated as a lemma:

Lemma III : The two defects in the respective terms of the radiation vertex expansion, (5.22), due to a given neutral internal particle, cancel each other in the null zone. (The conditions for the radiation interference theorem are assumed to hold.)

The proof of Lemma III follows by noting that the sum of the two defects in (5.22) is proportional in the null zone to

$$D(p')\, j_{out}(p') + j_{in}(p) D(p) \qquad \text{(null zone)}\ , \qquad (7.17)$$

since the remaining factors in the $M_\gamma R$ products are the same. In (7.17) the subscript I is suppressed and the currents j refer to the vertices that the internal line has left and entered (and can be found along with the propagator D in Table I). Also, $p' = p - q$.

In fact, (7.17) can be seen to be zero an argument based on the radiation decomposition identity (Table I). From a consideration of the original photon coupling to the internal line or vertex (the left-hand-side), the decomposition $(D'j' + j\,D)/p\cdot q$ (the right-hand-side) must be regular at $p\cdot q = 0$ (Q_I factors out.). Thus the expression in (7.17) vanishes in the null zone, where we must have from (7.14) that

$$p\cdot q = p'\cdot q = 0 \quad \text{(null zone)} \tag{7.18}$$

for any neutral internal line p . [Cf.(7.4).]

The vanishing of (7.17) establishes the lemma and allows us to regard a neutral line as a short-circuit between two vertices, leaving a composite gauge-invariant vertex that could be used in a reorganized radiation vertex expansion. It is noticed that none of the restrictions on external vector particles is needed for a neutral internal vector line. The defects in (7.17) have been correctly calculated, even if the neutral internal particle is a photon. Another gauge - see (7.12)-may be used with the same result.

Let us illustrate Lemma III by explicit verification for the various cases and with the examples in Sec. V. First, note that (7.18) does not imply that p and p' must satisfy (7.5), in contrast to external particles, so that in general

$$p\cdot\varepsilon = p'\cdot\varepsilon \neq 0 \quad \text{(null zone)} \quad . \tag{7.19}$$

However, (7.18) does imply that

$$p^2 = p'^2 \qquad \text{(null zone)} \qquad , \tag{7.20}$$

and hence the propagator denominators can be ignored in the demonstrations to follow.

We now evaluate (7.17), for the different particles and interactions relevant to the interference theorem, showing it to be zero in each case. For a scalar with constant couplings, (7.17) is proportional to

$$p'\cdot\varepsilon - p\cdot\varepsilon = 0 \quad . \tag{7.21}$$

If the scalar particle has a single-derivative coupling, given by p_σ ($r_3 = g_{\beta\sigma}$ in Sec. V.D) in the absorbing vertex, the relevant expression is

$$p'_\sigma\, p'\cdot\varepsilon - p_\sigma p\cdot\varepsilon + \omega_\sigma{}^\alpha p_\alpha = -\, p\cdot q\varepsilon_\sigma = 0 \quad , \tag{7.22}$$

by (7.18). If we add the derivative coupling p'_τ to the emitting vertex as well, we find, similarly,

$$p'_\sigma(p'\cdot\varepsilon p'_\tau + \omega_\tau{}^\alpha p'_\alpha) + (-p_\sigma p.\varepsilon + \omega_\sigma{}^\alpha p_\alpha)p_\tau = 0 \quad . \tag{7.23}$$

An example of a neutral internal scalar particle can be constructed from Fig. 6 and (5.4). We set $Q_1 = Q_4$, $Q_2 = Q_3$ so that $Q_5 = 0$ and the null zone corresponds to the subsequent limit $p_1\cdot q \to p_4\cdot q, p_2\cdot q \to p_3\cdot q$. The amplitude in (5.4) is zero in this null zone, but only by a cancellation between the square brackets. [A careful examination shows that (6.15) is not zero in this limit.]

A neutral Dirac internal line version of (7.7), with constant couplings, is proportional to

$$(\not{p}' + m)(p'\cdot\varepsilon + \tfrac{1}{4}\,[\not{\varepsilon},\not{q}]) - (p\cdot\varepsilon + \tfrac{1}{4}\,[\not{\varepsilon},\not{q}])(\not{p} + m) = 0 \ , \tag{7.24}$$

by (7.18-19). An example of this is found in the null zone cancellation, for $Q_6 = 0$ in Fig. 9, between the fermion defects in (5.48)-(5.49) and (5.52) - (5.53).

In the neutral-vector, internal-line case we reduce (7.17) to

$$P_{\gamma\beta}(p')(p'\cdot\varepsilon g^{\beta}_{\delta} + \omega_{\delta}{}^{\beta}) + (-p\cdot\varepsilon g^{\alpha}_{\gamma} + \omega_{\gamma}{}^{\alpha})P_{\alpha\delta}(p) = 0 \quad . \tag{7.25}$$

The cancellation for $Q_3 = 0$, in the example of Fig. 9, between the vector defects in (5.44)-(5.45) and (5.48)-(5.49) illustrates this case.

In the case where a neutral vector is emitted by a constant coupling and absorbed by a derivative coupling p_σ, (7.17) reduces, via (7.18) and (7.25), to

$$p'_\sigma P_{\gamma\beta}(p')(p'\cdot\varepsilon g^{\beta}_{\delta} + \omega_{\delta}{}^{\beta}) + p_\sigma(-p\cdot\varepsilon g^{\alpha}_{\gamma} + \omega_{\gamma}{}^{\alpha})P_{\alpha\delta}(p)$$

$$+ \omega_{\sigma}{}^{\alpha} p_\alpha P_{\gamma\delta}(p) - q_\sigma \omega_{\gamma}{}^{\alpha} P_{\alpha\delta}(p) = 0 \quad . \tag{7.26}$$

The last two terms, on the left-hand side, are the contact and momentum-shift-times-spin-current terms, respectively, and serve to promote p_σ to p'_σ in the second term. Derivative couplings at both ends go similarly.

VIII. EXTENSIONS TO NONGAUGE INTERACTIONS AND CLOSED LOOPS : A LOW-ENERGY THEOREM

We now consider interactions more general than those defined as gauge theoretic. The additional interactions involve first or higher-order derivatives of Dirac and vector fields (other than the Yang-Mills form) and/or second or higher-order derivatives of scalar fields. If we also allow closed loops, the source graphs can now be entirely arbitrary.

On the basis of what we have learned from the null zone cancellation for tree graphs, we present a low-energy theorem for <u>any</u> source graph in subsection A, followed by a simple example in B. The role of closed loops in radiation and a category of source graphs with closed loops for which the radiation interference theorem still holds are both discussed in C.

A. A low-energy theorem

The following theorem is a corollary of the radiation interference theorem:

<u>Null zone low-energy theorem</u> : For any source graph S_G, with $g = 2$ external legs, the radiation amplitude can be written as[37]

$$\mathcal{M}_\gamma(S_G) = M_\gamma(S_G) + O(q) \quad , \tag{8.1}$$

where M_γ satisfies the radiation interference theorem,

$$M_\gamma = 0 \qquad \text{(null zone)} \quad , \tag{8.2}$$

and has a radiation representation. See (2.9-10).

This theorem can be understood as the union of the standard low-energy theorem for bremsstrahlung[38,39] and the radiation interference theorem. In the standard low-energy theorem expansion for a given reaction the leading (infrared) term (3.18) vanishes in the null zone; the next-order (spin and contact) term also vanishes in the null zone provided that $g = 2$ for the external particles.

We define an effective tree graph substructure of S_G by the result obtained when all closed loops are contracted to points. We then consider an effective vertex radiation amplitude

$$M_\gamma(V_G) = \sum \frac{Q_i J_i}{p_i \cdot q} \tag{8.3}$$

in direct correspondence with (6.1). The infrared (linearly divergent) terms in M_γ correspond to convection terms in the effective currents J_i which are zeroth-order in q and which cancel in the sum, $\sum J_i$, by virtue of momentum conservation. The zeroth-order terms in M_γ correspond to the first-order spin and contact terms in J_i; these first-order terms cancel in the same sum by Lorentz invariance, provided that the photon couplings to the fixed lines in the effective tree graph correspond to $g = 2$. Since there is no general mechanism for the cancellation of higher powers of q, (6.2b) is then replaced by

$$\sum J_i = O(q^2) \quad . \tag{8.4}$$

The nonvanishing right-hand-side of (8.4) is the result of nongauge derivative couplings and closed loops.

The contact currents associated with the nongauge couplings and the closed-loop graphs[40] are straightforward to determine. The term that is linear in q in the expansion of the radiation graph where the photon is attached to an exterior leg of the closed loop or to a leg connected with a derivative coupling yields the momentum-shift part of the contact current. The seagull can be derived by requiring gauge invariance for both cases. (Alternatively, for the closed loop, the linear term from the graph where the photon is attached to the loop itself yields the seagull.) See Sec. V.D.

Although the external spinning particles are required to have $g = 2$, anomalous moments for internal particles contribute only at the $O(q)$ level, according to (5.23) and (5.34). Therefore, the internal lines need not have their photon couplings restricted to $g = 2$ in order for the null zone low-energy theorem to hold. For example, in the Dirac decomposition (5.21) internal $g \neq 2$ corrections correspond to quadratic terms in the numerators.

The zeroth-order and first-order terms in the J_i, which sum to zero, serve to define M_γ in the statement of the null zone low-energy theorem. It follows from Sec. VI that M_γ has a radiation representation. The quadratic terms associated with the Yang-Mills source vertex, which are the only higher-order terms in gauge-theoretic interactions and which cancel cyclically, could be included either in M_γ or in the $O(q)$ remainder of (8.1). This ambiguity shows that the null zone low-energy theorem is not equivalent to the radiation interference theorem, but is more properly called its corollary. On the other hand, the content of the full radiation interference theorem is the remark that the $O(q)$ terms in (8.1) are zero for gauge-theoretic couplings and tree graphs.

B. Example of the theorem

We first derive the standard low-energy theorem for the $n = 3$ radiative decay, $1 \to 2 + 3 + \gamma$, where the charged particles are all scalars. The amplitude, illustrated in Fig. 10, separates into external and internal radiative parts,

$$M_\gamma(\text{Fig. 10}) = M^{\text{ext}}(q) + M^{\text{int}}(q) \quad . \tag{8.5}$$

If $D(m_1^2, m_2^2, m_3^2)$ is the amplitude for the source decay, $1 \to 2 + 3$, then

$$\mathcal{M}^{ext}(q) = -\frac{Q_1}{p_1\cdot q}\, p_1\cdot\varepsilon\, D((p_1-q)^2, m_2^2, m_3^2)$$

$$+\frac{Q_2}{p_2\cdot q}\, p_2\cdot\varepsilon\, D(m_1^2,(p_2+q)^2,m_3^2) + \frac{Q_2}{p_3\cdot q}\, p_3\cdot\varepsilon\, D(m_1^2,m_2^2(p_3+q)^2) \ . \qquad (8.6)$$

The expansion of (8.6) in q leads to

$$\mathcal{M}^{ext}(q) = M_\gamma + \Delta\mathcal{M} + 0(q) \ , \qquad (8.7)$$

where

$$M_\gamma = (-\frac{Q_1}{p_1\cdot q}\, p_1\cdot\varepsilon + \frac{Q_2}{p_2\cdot q}\, p_2\cdot\varepsilon + \frac{Q_3}{p_3\cdot q}\, p_3\cdot\varepsilon)\ D(m_1^2,m_2^2,m_3^2) \quad , \qquad (8.8)$$

$$\Delta\mathcal{M} = 2(Q_1p_1\cdot\varepsilon\,\frac{\partial}{\partial m_1^2} + Q_2p_2\cdot\varepsilon\,\frac{\partial}{\partial m_2^2} + Q_3p_3\cdot\varepsilon\,\frac{\partial}{\partial m_3^2})\ D(m_1^2,m_2^2,m_3^2) \ . \qquad (8.9)$$

$\mathcal{M}^{int}$ is infrared convergent and can be expanded as

$$\mathcal{M}^{int}(q) = \mathcal{M}^{int}(0) + 0(q) \quad . \qquad (8.10)$$

In order to proceed further, we may follow either the approach of Ref. 39 or of Sec. V.D. The former approach centers on the observation that if $q^\mu f_\mu = 0(q^2)$ for arbitrary q, and if f_μ is independent of q, then $f_\mu = 0$. In our particular case, such an f_μ can be defined by

$$\varepsilon^\mu f_\mu \equiv \Delta\mathcal{M} + \mathcal{M}^{int}(0) \quad , \qquad (8.11)$$

because M_γ is separately gauge invariant. Therefore,

$$\mathcal{M}^{int}(0) = -\Delta\mathcal{M} \quad , \qquad (8.12)$$

so that

$$\mathcal{M}_\gamma(\text{Fig. } 10) = M_\gamma + 0(q) \quad . \qquad (8.13)$$

The approach of Sec. V.D is to relate ΔM in (8.9) and $M^{int}(0)$ in (8.10) to the momentum-shift and seagull terms, respectively, in the contact current (5.39). The vanishing of the contact current follows from the identity $p_i^{\mu} \omega_{\mu\nu} p_i^{\nu} = 0$ and corresponds to (8.12). [For the external leg i, $p = r = p_i$ in (5.38).]

As suggested by the notation we have used, the explicit (infrared) leading term in (8.13), M_{γ}, given by (8.8), vanishes in the null zone. The remainder, which does not vanish in the null zone in general, is $O(q)$. The statement of the null zone low-energy theorem has thus been verified. In a more complicated case where spin currents lead to zeroth-order terms in (8.13), these terms can also be incorporated into M_{γ} provided that $g = 2$ holds for the external particles with spin.

We note that the gauge-invariant radiation vertex expansion is useful in the general construction of low-energy theorems. In particular, it is well suited for dealing with the complications arising from cancellations between the two ends of a fixed internal line, from the effective-tree organization of graphs with closed loops, and from the definition[37] of $O(q)$.

The null zone low-energy theorem leads to a larger range of experimental tests, since we do not have to restrict ourselves to perturbative tree graphs. Some of these possibilities are proposed in the conclusions, Sec. XI.

C. Closed loops

The existence of amplitude zeros, which is central to the radiation interference theorem, at first sight may appear to violate the uncertainty principle. We do not expect, quantum mechanically, to find an exact cancellation in the interference among the various radiators at a specific point in momentum space, unless there is complete uncertainty in the particle positions. Indeed, the theorem refers only to the tree approximation where the radiation is controlled by the classical currents of plane wave states; $\hbar$ corrections from closed loops which provide coordinate correlations are expected to fill in the radiation amplitude zeros. In this respect, radiation zeros are in marked contrast to the exact amplitude zeros due to conservation laws such as angular momentum.

The absence of a radiation zero for particles with $g \neq 2$ (see Sec. V) is an example which can be attributed to quantum effects inasmuch as closed-loop radiative corrections give rise to anomalous magnetic moments. (In fact, the basic content of the Drell-Hearn-Gerasimov sum rule[41] is that deviations from $g = 2$ must be due to internal excitations.)

We recall from (8.1), that a violation of the interference theorem appears as an $O(q)$ contribution that has no radiation zero. In this context, the decay $1 \rightarrow 2 + \gamma$ provides a simple but instructive example (cf. Sec. IV.C). A physical $n = 2$ decay automatically satisfies the null zone condition so that M_γ vanishes identically. However, closed loops and nongauge couplings must lead to nonvanishing $O(q)$ contributions, unless another mechanism intervenes. Indeed, closed-loop amplitudes[42] for $\mu \rightarrow e\gamma$ do not vanish and are $O(q)$, when lepton number is not conserved. Although the $n = 2$ decay amplitude is identically zero to all orders for scalar particles 1 and 2, this is due to angular momentum conservation and the vanishing of its higher-order corrections can not be interpreted as a radiation zero.

The existence and position of a radiation zero does not depend on the spin of the external (or internal) particles and, moreover, does not depend on masses, charges, and momenta except in the $Q/p\cdot q$ combinations allowed by the null zone condition (4.1). By changing these parameters, one may test for a radiation zero. In the case of the $n = 2$ decay, adding spin eliminates the "angular-momentum" zero. As another example, the general amplitude, including closed loops, for the electron bremstrahlung reaction (4.6) would vanish by an angular momentum argument in the null zone (4.1), if the electrons were identical scalar bosons. Adding spin removes the angular-momentum zero in the high-order closed-loop amplitudes. On the other hand, adding closed loops removes the radiation zero, in general.

The previous remarks suggest two categories of closed-loop amplitudes for which there are amplitude zeros in the null radiation zone:

Category 0: This is the trivial class where the amplitude and its higher-order corrections vanish in the null zone because an additional mechanism, such as angular momentum conservation is operative in a subregion of the null zone for certain charge, mass, and spin assignments. Such mechanisms may be deactivated by changing the assignments or moving to another part of the null zone. Such closed-loop amplitude zeros are not radiation zeros.

Category 1: This is the class of source closed loops that produce no correlations or corrections to $g = 2$. We have in mind scalar self-energies, which can be included to all orders (see Sec. V.A), and "neutral" closed loops. If a closed loop is completely neutral (meaning there are no photon couplings to its internal lines with no charge transferred to it by external particles at any of its "external" vertices) and if the loop can be factorized so as to leave a Lorentz-invariant tree structure in the remainder, then the null zone cancellation can proceed according to that tree structure. It is noted that, if Δp_i is the momentum transfer to a neutral loop through its i^{th} neutral leg, $\Delta p_i \cdot \Delta p_j$ is invariant under photon emission from external lines, since $\Delta p_1 \cdot q = 0$ in the null zone. See the related remarks in Sec. VII.C.

Box graphs are closed loops that produce correlations. Self-energy source loops for spinning particles lead to $g \neq 2$. These examples do not belong to category 1.

IX. PHOTON COUPLING : POINCARÉ TRANSFORMATIONS AND BMT

We have established the relationship between the form of the spin and contact currents in gauge theories and the first-order terms of a universal homogeneous Lorentz transformation. In addition, the cancellation of the convection currents in the null zone depends on momentum conservation, implying a relationship to translational invariance. In this section we unify these ideas in terms of the Poincaré group of transformations, relating the currents to the appropriate generators. We also find an important connection between the BMT equations and the null zone cancellations, where a universal transformation also arises.

A. Poincaré transformations

Let us recall the universal Lorentz transformation (5.12),

$$\Lambda_{\mu\nu} = g_{\mu\nu} + \lambda\omega_{\mu\nu} \quad , \tag{9.1}$$

in first order, where λ has the dimension of a length and represents the freedom in normalization. We may rewrite (5.13),

$$\lambda\omega_{\mu\nu} = q_\mu d_\nu - d_\mu q_\nu \quad , \tag{9.2}$$

in terms of the space-like four-vector

$$d_\mu \equiv \lambda\, \varepsilon_\mu \quad , \tag{9.3}$$

($d^2 = -\lambda^2$) which is transverse to q and has the same dimension as λ. The generalization of (9.1) to finite λ is $\exp(\lambda\omega)$ or

$$\Lambda_{\mu\nu} = g_{\mu\nu} + \lambda\omega_{\mu\nu} + \frac{\lambda^2}{2}\, q_\mu q_\nu \tag{9.4}$$

(It is always assumed that $q^2 = q\cdot\varepsilon = 0$.) Since

$$\Lambda^\mu{}_\alpha g_{\mu\nu} \Lambda^\nu{}_\beta = g_{\alpha\beta} \quad ,$$

$$\Lambda^\mu{}_\nu q^\nu = q^\mu \quad , \tag{9.5}$$

we see that the $\Lambda_{\mu\nu}$ form an Abelian subgroup of the little group E_2 defined by q.[43] We also see that Λ generates gauge transformations on the polarization vector ε,

$$\Lambda^{\mu}{}_{\nu}\varepsilon^{\nu} = \varepsilon^{\mu} - \lambda q^{\mu} \quad . \tag{9.6}$$

An important result of Sec. V is that the spin and contact currents can be written in terms of the universal first-order term in the Lorentz transformation (9.1). In addition, we observe that the convection current $p\cdot\varepsilon$ can be understood as the universal first-order term in the translation ($e^{ip\cdot a} \to 1 + ip\cdot a$) in the direction ε. Since the relative normalization among the currents is fixed, we must have $a = d$. The length d_{μ} then appears universally in the generator (9.2) for the spin and contact currents and as the displacement for the convection currents. These circumstances suggest that we consider the full Poincaré transformation $P = \{d,\Lambda\}$:

$$x' = \Lambda x + d \quad . \tag{9.7}$$

Each of the current constributions in Table I can be expressed universally in terms of the first-order Poincaré transformation P acting on the particle wave functions. (The internal currents are understood, via the decomposition identities, as transformations on bilinear wave functions.)

The vanishing in the null zone of the radiation amplitude for tree diagrams in gauge theory can thus be described in terms of Poincaré symmetry: The convection current cancellation by translational invariance and the spin and contact current cancellation by Lorentz invariance.[44] (The Yang-Mills cancellation involves additional symmetry.)

To explore further the connection to Poincaré symmetry, consider the electromagnetic current J^{μ} in lowest order. The current has a Gordon decomposition[45] into the separately conserved convection and spin currents,

$$J^{\mu}_{conv} = \sum_j iQ_j \tilde{\psi}_j \overleftrightarrow{\partial}^{\mu} \psi_j \qquad , \tag{9.8}$$

$$J^{\mu}_{spin} = \sum_j 2iQ_j \partial_\nu (\tilde{\psi}_j S^{\mu\nu} \psi_j) \qquad , \tag{9.9}$$

where the spin indices of the fields ψ have been suppressed and where $\tilde{\psi} \equiv \psi^+$ or $\bar{\psi}/2m$ as the case may be. The summations in (9.8) and (9.9) are over all charged particle fields.

The spin tensor in (9.9) is

$$S_{\mu\nu} = \begin{cases} 0 \quad , \quad \text{scalar,} \\ -\frac{i}{2}\,\sigma_{\mu\nu} \;, \quad \text{Dirac} \;, \\ i(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}) \;, \text{ vector} \;. \end{cases} \tag{9.10}$$

The indices σ,ρ in the vector case are those of the fields in (9.9).

The spin-current Lagrangian, $-J^{\mu}_{spin}A_\mu$, corresponds to the interaction Hamiltonian,

$$H_{int} = \sum_j iQ_j \tilde{\psi}_j S^{\mu\nu} \psi_j F_{\mu\nu} \qquad , \tag{9.11}$$

which, for the $S_{\mu\nu}$ given by (9.10), implies the gyromagnetic value, $g = 2$, for each particle with spin. Therefore, (9.10), which is also the set of matrix representations of the generators of Lorentz transformations on spins 0, ½, and 1, respectively, exhibits a direct connection between the spin current and the Lorentz transformation of the fields,[46] but <u>only for $g = 2$</u>.

B. The BMT analysis and the null zone

Since the radiation amplitude is linear in the photon field, the correspondence principle implies that there should be a classical counterpart for the relationship of $g = 2$ to the universal Lorentz transformation found in Sec. V. Let us investigate this point for a classical particle with spin moving in a slowly varying external electromagnetic field $F^{\mu\nu}$. Our neglect henceforth of forces dependent upon the gradients of the fields is consistent with the fact that the null zone cancellation involves only the first two orders in q .

The motion of a particle with charge Q and mass m moving in $F^{\mu\nu}$ is described by[47]

$$\frac{du^{\mu}}{d\tau} = \frac{Q}{m} F^{\mu\nu} u_{\nu} \quad , \tag{9.12}$$

where u is the four-velocity and τ is the proper time. The BMT equation for the four-polarization s of the particle is[47,48]

$$\frac{ds^{\mu}}{d\tau} = \frac{Q}{m} \frac{g}{2} F^{\mu\nu} s_{\nu} + \frac{Q}{m} \left(\frac{g}{2} - 1\right) u^{\mu} s_{\lambda} F^{\lambda\nu} u_{\nu} \quad , \tag{9.13}$$

with gyromagnetic ratio g . A significant and well-known feature of (9.12) and (9.13) is that, for $g = 2$, the changes in u and s in time $d\tau$ can be described in terms of the same infinitesimal Lorentz transformation,

$$\Lambda_{\mu\nu} = g_{\mu\nu} + \frac{Q}{m} F_{\mu\nu} \, d\tau \quad . \tag{9.14}$$

Consequently, in proper time $d\tau$, the orbital and precessional frequencies of the particle are identical.

What is of interest is the situation involving a system of particles moving in $F^{\mu\nu}$. In order to compare the Lorentz transformation (9.14) for each particle we refer to a single common observer at a (retarded) time t , which is related to the particle times t' by[49]

$$dt = dt' \, (1-\hat{n}\cdot\vec{v}) = \frac{p\cdot n}{E} \, dt' \quad , \tag{9.15}$$

Here $\vec{v}$ (E) is the velocity (energy) of a given particle and $\hat{n}$ is the unit vector from this particle to the (distant) observer such that $n \equiv (1,\hat{n})$

is a light-like 4-vector proportional to the radiation wave 4-vector.

From (9.14), (9.15), and $dt' = Ed\tau/m$,

$$d\Lambda_{\mu\nu} = \frac{Q}{p \cdot n} F_{\mu\nu}\, dt \quad . \tag{9.16}$$

At a given time, all particles with identical $Q/p\cdot n$ and with $g = 2$ are observed to have the identical response to the presence of a constant external field. The condition of identical $Q/p\cdot n$ is equivalent to the null zone condition since the photon energy can be scaled out of the equations in (4.1). (An initial particle simply corresponds to an earlier t' than does a final particle.) The first-order Lorentz transformation (9.16) can be compared to (9.1) by keeping in mind that $\omega_{\mu\nu}$ is the Fourier transform of the radiation counterpart to $F_{\mu\nu}$. [Also, the role of the translation in Poincaré transformations is implicit in the integration of (9.11).]

Thus all Lorentz invariants constructed of u_i, s_i and their derivatives, such as those that arise in the Lagrangian, are fixed in the time interval during which all $Q/p\cdot n$ are equal (and $g = 2$). [Equivalently, we may think of making an instantaneous Lorentz transformation which cancels (9.16).] In this sense, a system of particles in its null zone experiences no linear response to a slowly varying external field. If we now identify $F_{\mu\nu}$ with the radiation field, then this result corresponds to the radiation interference theorem.

X. EXTENSIONS TO RADIATION OF OTHER GAUGE BOSONS

In this section we investigate the extent to which the radiation interference theorem applies to the emission/absorption of other massless gauge bosons besides the photon. We also briefly discuss the emission of other kinds of particles, with different mass and spin.

A. Other gauge bosons

The radiation interference theorem, the radiation representation, and the associated corollaries can be proven for an arbitrary gauge group, G, where the role of the photon is assumed by the massless gauge boson(s), g, assigned to the adjoint representation of G. If the generalized "charges", calculated from the representations of G to which the particles belong, are conserved, then it is easy to adapt the previous line of proof. Generalizing from the U(1) case considered previously, the current for g emission has a dual connection to both internal-group (G) transformations and space-time Poincaré-group transformations and the invariance under each group can be exploited.

Our task is facilitated by the results[5] of Goebel, et al., where the four-body amplitude zero has been related to factorization for general G, and where useful notation is introduced. The analysis of Ref. 5 is limited to an n = 3 vertex source graph, in our terminology, and our first step is to generalize their work to an arbitrary n-vertex source graph.

We assume that g has local gauge couplings to all other particles (possibly including more gauge bosons g), which belong to the various representations of G and whose couplings are invariant under G. If we use Feynman rules in factored form, the n-vertex source graph can be written as a product of an internal-group factor and a space-time factor,

$$S_V = \Gamma_{a_1a_2\cdots a_n} V(p_1,p_2,\ldots,p_n), \tag{10.1}$$

which are invariant under G and Lorentz transformations, respectively. The space-time factor V is identical to that for the previous special case of the photon, G = U(1). The internal-group factor Γ is the Clebsch-Gordon coefficient of G for the n-particle coupling, labeled by the internal symmetry indices a_i which refer to the various representations and which are tied together in an invariant fashion

The complete g-emission "radiation" amplitude, whose source graph is given by (10.1), has the same general structure as (6.1) with the same space-time current J_i ,

$$M_g = \sum_1^n \frac{Q_i^g J_i}{p_i \cdot q} \quad . \tag{10.2}$$

The gauge-boson couplings,

$$Q_i^g = \Gamma_{a_1a_2\cdots a_{i-1}ba_{i+1}\cdots a_n} \Gamma_{\bar{b}aa_i} \quad , \tag{10.3}$$

where a sum over b is understood, represent a generalization of the U(1) charge. Here $\Gamma_{\bar{b}aa_i}$ is the Clebsch-Gordan coefficient for the 3-vertex which couples an incoming particle i, the gauge boson g (with index a), and an outgoing particle with index b .

Common factors,

$$\frac{Q_i^g}{p_i \cdot q} = \frac{Q_j^g}{p_j \cdot q} \quad , \quad \text{all } i, \text{ any } j, \tag{10.4}$$

lead to the vanishing of the amplitude in (10.2) in familiar fashion. In addition, the generalized charges also sum to zero [cf. (6.2a)],

$$\sum_{i=1} \delta_i Q_i^g = 0 \ , \tag{10.5}$$

since the fact that g is in the adjoint representation of G implies that $\Gamma_{\bar{b}aa_i}$ refers to a matrix representation of the corresponding generator in G. [Owing to the G-invariance of the source graph, the complete sum of the charges in (10.2) is zero, corresponding to a vanishing total commutator and yielding (10.5).] Therefore, an n-2 double-difference radiation representation can also be obtained for (10.2), with the qualifications concerning any derivative couplings in J_i the same as in the photon case.

We next demonstrate that the above results can be extended to tree graphs with internal lines. The emission of g from any given internal line involves the G-space factor,

$$\ldots\Gamma_b^L \Gamma_{\bar{b}ac} \Gamma_{\bar{c}}^R \ldots \ , \tag{10.6}$$

in which the "left" vertex, with coefficient Γ_b^L, is connected to the "right" vertex, $\Gamma_{\bar{c}}^R$ by the original internal line, $\delta_{\bar{b}c}$, in the source graph. The other source-graph indices and Clebsch-Gordan factors are suppressed in (10.6). The remaining task is to generalize the radiation decomposition identity to include (10.6).

Referring to the schematic in Fig. 5, we associate $\Gamma_{\bar{b}ac}$ first with $\Gamma_{\bar{c}}^R$ and then with Γ_b^L, respectively, in the corresponding emission terms of the decomposition identity so that there is a complete set of conserved charges, analogous to (10.3), associated with each source vertex. This thus gives a generalized gauge-invariant radiation vertex expansion. The radiation interference theorem, its corollaries, the radiation representation, and the other photon results all generalize to G with the replacement of Q_i by Q_i^g.

Dongpei[6] has worked out SU(2) and SU(3) examples corresponding to $n = 3$ that provide useful illustrations of the above results. Suppose that the 3-vertex source graph is the spinor-spinor-vector coupling, $\bar{\psi}\gamma_\mu T_a \psi V_a^\mu$, where the Dirac particles, 1 and 2, and the vector particle, 3, belong to the fundamental and adjoint SU(N) representations, respectively. Then the constraint, $Q_1^g/p_1 \cdot q = Q_2^g/p_2 \cdot q$, becomes

$$\frac{(T_a T_b)_{ij}}{p_1 \cdot q} = -\frac{(T_b T_a)_{ji}}{p_2 \cdot q} \quad . \tag{10.7}$$

Solutions for equal masses are given in Ref. 6.

There is a practical limitation to the observation of certain non-Abelian radiation zeros which has been noted previously in the 3-vertex case.[6] In the case of QCD, the gluon is coupled to (presumably) unobservable color charges. Therefore, the color-singlet physical states are connected to quark and gluon particles only through color averaging and summing. Since their positions depend on the charges, the amplitude zeros are smeared out in the physical cross sections. We emphasize, however, that the radiation representation for the gluon amplitudes remains valid.

B. Other spins and masses

We have noted that the vector character of the gauge boson is essential to the association of the currents with Poincaré transformations. Thus, spin-one particles and Lorentz invariance appear to go hand-in-hand in the crucial null zone cancellations, a relationship that is absent for the emission/absorption of particles with other spins.

Nevertheless, other spins and relationships should be investigated. We have in mind graviton emission and Riemann invariance, as well as superfield emission and supersymmetry. These questions are not addressed in this paper, but the search for currents that satisfy analogous dualities may be fruitful.

Finally we consider whether the results can be extended to (Abelian or non-Abelian) vector gauge bosons with mass. Let us consider $q^2 \neq 0$, addressing the two cases where the radiated boson is virtual (e.g., lepton scattering and e^+e^- annihilation) and where it is real with nonzero mass (e.g., Z^o production in electroweak theory). We note that $q \cdot \varepsilon = 0$ still holds in both, but $q \cdot \varepsilon$ should be retained in order to check gauge invariance. (In the virtual case we assume that ε^μ represents a conserved current source.)

We now reconsider the calculations of the currents for $q^2 \neq 0$. We find that the convection and Dirac spin factors in Table I for both external and decomposition-identity emission factors are changed only by the replacement

$$\frac{Q}{p \cdot q} \to \frac{Q}{p \cdot q \pm \frac{1}{2} q^2} \tag{10.8}$$

for outgoing (+) or incoming (-) particles. (Strictly, the gauge-invariant convection current is $\pm p \cdot \varepsilon + \frac{1}{2} q \cdot \varepsilon$.) The vector-particle spin factor requires two changes, (10.8) and

$$\omega_{\mu\nu} \to \omega_{\mu\nu} + \frac{1}{2m^2} (p \pm q)_\mu \left[q^2 \varepsilon_\nu - q \cdot \varepsilon \, q_\nu \right] \quad , \tag{10.9}$$

where $p \pm q$ is the momentum of the vector particle between the source vertex and the emission. The change in (10.9) does not contribute in the event that the vector particle is itself coupled to a conserved current. However, if gauge invariance requires seagull contributions, the contact current of Table I is significantly altered,

$$\omega^{\beta\mu} p_\mu \to \omega^{\beta\mu} p_\mu \mp \frac{1}{2} \left[q^2 \varepsilon^\beta - (q \cdot \varepsilon) q^\beta \right] \quad , \tag{10.10}$$

Evidently, Lorentz invariance does not also imply the cancellation of the new term, appearing in (10.10), in the ΣJ_i sum.

Another difference is that the new factors (10.8) cannot be equal in the physical region, in general.[50] The absence of physical null zones corresponds to the absence of asymtotic radiation fields (r^{-1} behavior). Furthermore, there is no analog to (6.2c) for the denominators, unless the number of particles is unchanged during the collision, so that we cannot generally reduce the number of differences from n-1 to n-2. Despite these remarks, we can again write a radiation representation, in terms of n-1 (or n-2) differences or products of differences, depending on whether (6.2a) and (6.2b) are valid. A simplified, gauge-invariant expansion follows. In the case of broken gauge symmetries such as the SU(2) x U(1) electroweak theory, the radiation interference theorem holds in the approximation at high energies where masses are neglected. In this connection, see the angular distributions for the reaction $q\bar{q} \to W^{\pm}Z^{o}$ in Ref. 3.

XV. SUMMARY AND FUTURE DIRECTIONS

This paper contains an elaboration of the details underlying Ref. 1 as well as new results about the occurrence and implications of zeros in gauge-boson amplitudes. In this section we summarize our principal results. We also discuss what appear to us to be important and interesting future directions.

A. Summary

We have found that there are zeros in every tree photon amplitude, provided only that any derivative couplings involved are of renormalizable (minimal) form or are products of such forms. Gauge theories are just such theories. The positions of the zeros depend only on the external charges and momenta through the ratios $Q/p\cdot q$, are independent of spin, and may lie in both physical and unphysical regions. This result can be extended to other massless-gauge-boson tree amplitudes.

We have introduced a useful radiation vertex expansion, (2.5) or (5.22), $\Sigma M_\gamma(V_G)R(V_G)$. The complete set of Feynman diagrams for the photon (or other massless gauge boson) attachments to the source tree graph T_G is thereby rewritten in terms of radiation vertex amplitudes $M_\gamma(V_G)$, each of which is a sum $\Sigma QJ/p\cdot q$ over photon attachments to V_G calculated as if all vertex legs were external. Consequently, each $M_\gamma(V_G)$ is separately gauge invariant (under electromagnetic gauge transformations). The radiation decomposition identity is instrumental in effecting this reorganization. (cf. Table I.)

The general form $\Sigma QJ/p\cdot q$ for the radiation vertex amplitude clearly exhibits the basic algebra leading to the radiation interference theorem and its complement. If $Q/p\cdot q$ $(J/p\cdot q)$ is the same for all legs of the vertex, and if $\Sigma J = 0$ ($\Sigma Q = 0$), then $M_\gamma(V_G) = 0$. (For simplicity, we have taken[8] all particles as outgoing.) As a consequence of $\Sigma J = \Sigma Q = \Sigma p\cdot q = 0$, $M_\gamma(V_G)$ can be rewritten in the form $\Sigma p\cdot q(Q/p\cdot q - A)(J/p\cdot q - B)$ for any A,B. The radiation representation (6.6) is obtained by choosing A(B) to be a particular factor $Q/p\cdot q$ ($J/p\cdot q$). The two interference theorems are made explicit with such a representation.

The fundamental relation underlying the radiation interference theorem is $\Sigma J = 0$, which might be called the Poincaré-Yang-Mills sum rule or conservation law. Noting the conservation of charge, $\Sigma Q = 0$, we see a dual role for the electromagnetic (or other gauge group) current: The current generates transformations in the internal gauge-group space and also, in effect, generates transformations in space-time. (After factoring out $Q/p\cdot q$, the convective current effectively generates a universal displacement, the spin current effectively generates a universal space-time Lorentz transformation of its associated wave function, and the contact current effectively generates the same universal Lorentz transformation of its associated derivative coupling. See Table I.) In this way we can view the massless gauge boson as characteristic of the adjoint representation of both the internal gauge group and the relevant little group, whose attachment generates the product of the first-order gauge and Poincaré (displacement and Lorentz) transformations, provided we have the prescribed derivative couplings. Poincaré and Yang-Mills symmetries[51] are thus responsible for $\Sigma J = 0$ which gives the null zone cancellation.

The existence of the radiation zeros has important algebraic significance, whether or not they occur in physical regions. As an added benefit, it is not difficult to find realistic reactions whose null zones overlap with the physical region. A physical null zone theorem has been given which states that if particles have the same Q/m ratios (more generally, the common value of Q/m for the initial state may be different from that for the final state) then we can always find, at any c.m. energy, physical regions where the radiation zeros occur (i.e., where all $Q/p\cdot q$ are equal). The Q/m restriction can be relaxed for any particle that is massless; we note that the physical null zone is generally smaller for particles with mass.

We have also studied physical null zone limits for more general Q,m values in the $n=3$ case and for equal Q/m in $n=4$.[52] In such studies, we have used an amusing identity, (4.2), based on the simple remark that $(a+b)/(A+B) = a/A$ if $a/A = b/B$. This is also used in the reduction of the number of independent $Q/p\cdot q$ factors and in the demonstration that the internal $Q_I/p_I\cdot q$ factors are equal to the external $Q/p\cdot q$ factors in the null zone.

We have shown that the radiation interference theorem applies in the case where there are additional neutral external particles provided that these additional particles are massless (and couple to conserved currents if they have spin 1). The null zone requirement for the massless external neutral particle r $(Q_r = 0)$ is that it must travel in the same direction as the photon $(p_r\cdot q = 0)$, which implies that $J_r = 0$. The analogous remark for the complementary interference theorem is that $J_r = 0$ would require $Q_r = 0$.

Neutral <u>internal</u> particles, however, do not have such restrictions. In the example, $e^-e^- \to e^-e^-\gamma$, discussed in Sec. IV, the electron emission currents cancel in the null zone across the neutral internal photon line in each of the crossed and uncrossed source graphs individually.

We note that the radiation representation applies independently of the values for $Q/p\cdot q$. In particular, it applies irrespectively of whether there are neutral particles or whether such particles satisfy the special criteria of masslessness and conserved-current couplings in the external case.

The radiation interference theorem is the statement that gauge-theoretic interactions preserve the classical (infrared) zeros in tree approximation. The null zone condition could just as well be defined as the condition under which there is complete destructive interference of the classical radiation patterns of the incoming and outgoing charged lines (the infrared limit). In the nonrelativistic limit, this corresponds to the well-known absence of electric dipole radiation for collisions involving particles with the same charge-to-mass ratio. The universal photon currents listed in Table I are the counterparts of the classical BMT equations for $g = 2$, according to the discussion in Sec. IX.

We have stressed the unique properties of radiation zeros, most notably the remarkable feature of spin independence, that distinguishes them from other amplitude zeros. The lowest-order differential cross sections for the various spin states in several reactions which include (1.11), $u\bar{d} \to W\gamma$, have been examined recently[53] for additional zeros. Besides the radiation zero, other zeros are also found but which depend on the polarization. Only the radiation zero is present in every helicity channel.

On the other hand, radiation zeros are generally destroyed by closed-loop (higher-order) corrections. The existence of these short-range quantum corrections can be anticipated from the uncertainty principle. One cannot expect exact amplitude zeros for subregions of angles and energies except in the violation of a conservation law. The special class of closed loops, where there are no correlations and no $g = 2$ corrections, is an exception. Thus, we can include certain neutral closed loops defined in Sec. VIII.

We can also include scalar self-energies in the source graph since the radiation decomposition identity is correct to all orders for scalar particles p, p'. (See Sec. V.) Indeed, in a recent study of scalar particles in the null zone[54] it is shown that first-order scalar bubbles preserve the radiation zero while a triangle source graph does not. In the context of our discussion, the former example introduces neither a correlation nor an anomalous moment, while the latter generates a correlation.

We have formulated a null zone low-energy theorem which is based on the fact that the radiation interference theorem can be applied to the leading terms of any expansion in photon momentum q. The infrared term, which is $O(q^{-1})$ and is analyzed in Sec. III, is guaranteed to vanish in the null zone for arbitrary amplitudes including closed loops and non-gauge-theoretic interactions. The $O(q^{0})$ term also vanishes in the null zone provided that the external particles have $g = 2$. Therefore, all low-energy theorems automatically separate out those (leading) terms that have radiation zeros. We have also presented a useful formalism for combining the study of low-energy theorems and the null zone by means of a generalized radiation vertex expansion for the effective tree structure of an arbitrary source graph.

B. Remarks

It is well-known that gauge theory couplings can be derived by imposing a unitarity constraint on the high-energy limit of tree amplitudes.[55] Since electromagnetic minimal couplings can also be inferred by the requirement that the radiation interference theorem hold, we seem to be building a bridge from the classical infrared limit to high energy behavior. We note also that the DHG sum rule for anomalous moments implies that we should have the gyromagnetic moment $g = 2$ for all spins at the tree level (classical limit), given a high-energy condition on the spin-flip Compton amplitude. The same conclusion follows for the existence of null radiation zones.

Furthermore, it has been suggested to us that the radiation interference theorem could possibly be stated directly in terms of renormalizability:[56] "The necessary and sufficient condition for a tree amplitude with one or more external massless gauge particles to have a zero independent of spin is that the model be renormalizable, where the renormalizability may be disguised by a Higgs mechanism or by heavy particles whose exchange looks like a point interaction (tree segments of zero length)." In this sense, the class of gauge-theoretic interactions, defined in Sec. II, may be called <u>quasi-renormalizable</u>.

The most striking experimental implication of the radiation zeros involves the original reaction, $q\bar{q} \to W\gamma$ in (1.11), which should be measurable[57] in future $p\bar{p} \to W\gamma X$ experiments at CERN and Fermilab. Although the actual external legs are integrally charged hadrons with anomalous moments, the high transverse momentum photon, recoiling against the W, couples in leading twist only to the hard-scattering subprocess; diagrams involving radiation from spectators, etc., are suppressed by powers of m^2/M_W^2 where m is the hadronic mass scale. In addition, there are quantum corrections from QCD loop diagrams that are of order $\alpha_s(M_W^2)/\pi$ by the standard renormalization group analysis and there is transverse momentum smearing from the hadronic wave functions and the gluon radiative corrections. To this accuracy, gauge theory couplings can be probed. The investigation of null zones in bremsstrahlung reactions such as hard quark scattering,

$q\,q \to q\,q\,\gamma$, or in radiative decays may give a measure of heavy quark and heavy lepton magnetic moments.

In principle, a measure of the neutrino mass m_ν can be found in the decay, $A \to B + \nu + \gamma$, since its null zone requires $m_\nu = 0$.[58] It has also been suggested that corrections to PCAC may be similarly studied.[59] In general, the deviations from zero in the null zone provide estimates of higher-order corrections [which must also be $O(q)$ by the null zone low-energy theorem] in any process, from the standard reactions such as $e^-e^- \to e^-e^-\gamma$ to exotic processes involving new particles.

The null zone condition can be applied very simply to composite particles with arbitrary spin and with collinear constituents i (momenta $p_i = x_i p$ in terms of the composite momentum p). This immediately applies to hadrons involved in hard scattering QCD processes. In the region where $x_i \propto Q_i$, the tree-graph approximation with gauge couplings for the constituents implies that the effective composite particle has the same $Q/p\cdot q$ factor as its constituents. Furthermore, the resultant effective current follows the description in Table I, corresponding to an effective gauge coupling for the composite. The null zone is preserved. More generally, we may use a composite picture to understand the null zone in any radiative reaction. Both the initial and final states can be considered to be composites with factors $Q_1/p_1\cdot q$ and $Q_2/p_2\cdot q$, respectively. Thus, in the null zone, we may view the reaction as equivalent to $1 \to 2 + \gamma$ whose tree amplitude vanishes for $Q_1/p_1\cdot q = Q_2/p_2\cdot q$, irrespective of the spin of the composites.

Another topic of theoretical interest is whether the radiation representation (6.6), when combined with the radiation vertex expansion (5.22), could be used to simplify cross section calculations. Recent calculations of radiative processes in QED and QCD have shown that the lowest-order unpolarized differential cross sections are generally very simple and factorize in final form.[60] We have verified that radiation zeros are present in these forms and are located in a single factor. For example, the reaction $e^+e^- \to e^+e^-\gamma$ has the same (unphysical) null zone as that of $e^+e^- \to \mu^+\mu^-\gamma$, when lepton masses are neglected. Indeed, we find a common factor in the two expressions for the differential cross sections of the two reactions, in which the radiation zeros reside. The symmetries inherent in the concept of radiation zeros can be instrumental in understanding the simplicity of the cross section forms obtained.

Finally, it is important to determine the extent to which currents in theories of higher spins such as supersymmetry play an analogous role. Do they also generate transformations in both internal and external spaces in the manner of the massless vector gauge boson currents? Will they also lead to equations which relate variables in both spaces like $Q_i/p_i \cdot q = Q_j/p_j \cdot q$?

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APPENDIX : PHYSICAL NULL ZONE

This appendix is addressed to the question of where the null radiation zone lies in phase space and, in particular, some details behind the equations of Sec. IV. We also describe an approach to many-particle null zones, including the theorem and its corollary that are stated in Sec. IV.E.

1. The n = 3 decay

We begin with the boundary limits for the decay, $1 \to 2+3+\gamma$. The lower (upper) limit on the range in (4.13) is derived from $p_3 \cdot q \geq 0$ $(E_2 \geq m_2)$. The range in (4.14) is obtained from

$$y^2(x+\mu_3^2) + yx(x+\mu_2^2+\mu_3^2-1)+\mu_2^2 x^2 \leq 0 \quad , \tag{A1}$$

which is a consequence of

$$q^2 = 0 = m_1^2 + m_2^2 + m_3^2 - 2m_1(E_2+E_3) + 2(E_2E_3 - \vec{p}_2 \cdot \vec{p}_3) \tag{A2}$$

and

$$(\vec{p}_2 \cdot \vec{p}_3)^2 \leq \vec{p}_2^{\,2} \, \vec{p}_3^{\,2} \quad . \tag{A3}$$

A physical null zone for the decay amplitude exists only if (A1) is satisfied for y given by (4.12):

$$Q^2(x+\mu_3^2) + Q(x+\mu_2^2+\mu_3^2-1)+\mu_2^2 \leq 0 \quad , \tag{A4}$$

in terms of the relative charge

$$Q \equiv \frac{Q_2}{Q_3} \quad . \tag{A5}$$

[Note that a factor x^2 has been divided out in obtaining (A4).]

Since $x \geq 0$, (A4) yields

$$Q^2\mu_3^2 + Q(\mu_2^2+\mu_3^2-1) + \mu_2^2 \leq 0 \quad . \tag{A6}$$

From (A6), or from (4.14),

$$Q_- \le Q \le Q_+ \quad ,$$

$$Q_\pm = \{1 - \mu_2^2 - \mu_3^2 \pm [(1 - \mu_2^2 - \mu_3^2)^2 - 4\mu_2^2\mu_3^2]^{\frac{1}{2}}\}(2\mu_3^2)^{-1} \quad . \tag{A7}$$

Therefore, given some masses m_2 and m_3, only those charges (A5) that lie in the range (A7) can lead to a physical null zone. The massless limit of (A7),

$$0 \le Q \le \infty \quad , \tag{A8}$$

confirms the existence of a physical null zone for all same-sign charges.

There is a broad range of physical possibilities allowed by (A7) for $\mu_i \neq 0$, as well. To see this, let us calculate the mass limits, for a given Q in the domain (A8), using (A6). In $m_2 - m_3$ space for a given ordering, we find

$$0 \le \mu_2^2 \le \frac{Q}{Q+1} < 1 \quad , \tag{A9a}$$

$$0 \le \mu_3^2 \le \frac{1}{Q+1} - \frac{\mu_2^2}{Q} \quad . \tag{A9b}$$

The inequality (A9b) is seen to be consistent with the basic mass inequality,

$$m_2 + m_3 \le m_1 \quad , \tag{A10}$$

since $(Q+1)^{-1} - \mu_2^2/Q \le (1-\mu_2)^2$. Inequality (A10) is implicit in (4.13) and is equivalent to the positivity condition on the radicand of (A7).

The nonrelativistic limit is the upper limit of (A10) with vanishing photon energy,

$$\mu_2 + \mu_3 = 1 \quad , \tag{A11}$$

From (A11) we infer that μ_3^2 takes its maximum value in (A9b) and therefore $\mu_3 = (Q+1)^{-1}$, $\mu_2 = Q/(Q+1)$, so that

$$\frac{Q_2}{m_2} = \frac{Q_3}{m_3} = \frac{Q_1}{m_1} \quad , \tag{A12}$$

where the last equality is derived as usual through mass and charge conservation and (4.4).

Suppose that we are given identical ratios,

$$\frac{Q_2}{m_2} = \frac{Q_3}{m_3} \quad , \tag{A13}$$

but not necessarily in the nonrelativistic limit. (They cannot also be equal to Q_1/m_1 , except in that limit.) If (A13) is valid then (A9) yields

$$\mu_2 = Q\ \mu_3 \ \leq \ \frac{Q}{Q+1} \quad . \tag{A14}$$

However, this is the same as what is implied by (A10) and (A13) alone. Thus, all values of m_2/m_3 consistent with (A13) and (A10) produce a null zone. We reiterate that (A13) does not imply (A12), but only that all $Q_i/p_i \cdot q$ can be equal. This is generalized by a theorem in Sec. IV.E .

2. The n = 3 scattering

We proceed to the two-body scattering, $1 + 2 \to 3 + \gamma$, discussed in Sec. IV.C. In terms of the initial c.m. speeds, v_1 and v_2 , (4.17) may be rewritten

$$\cos\theta = (Q_2/v_1 - Q_1/v_2)/Q_3 \quad . \tag{A15}$$

Confining (A15) to the physical region,

$$-1 \leq \cos\theta \leq 1 \quad , \tag{A16}$$

and defining a relative charge

$$Q \equiv \frac{Q_2}{Q_1} \quad , \tag{A17}$$

we obtain

$$\frac{v_2^{-1} - 1}{v_1^{-1} + 1} \leq Q \leq \frac{v_2^{-1} + 1}{v_1^{-1} - 1} \quad , \tag{A18}$$

for given v_i . On the other hand, we have

$$1 \leq v_1^{-1} \leq \infty \quad , \tag{A19a}$$

$$\max\,[1,\ Q(v_1^{-1} - 1) - 1] \leq v_2^{-1} \leq Q(v_1^{-1} + 1) + 1 \quad , \tag{A19b}$$

for a given Q . In the equal-mass case, (A19) reduces to

$$\frac{|Q-1|}{Q+1} \leq v \leq 1 \quad ,$$

$$v_1 = v_2 \equiv v \quad . \tag{A20}$$

We may check several limits of the above equations. First, the overall limit on Q governed by (A18) is

$$0 \leq Q \leq \infty \quad , \tag{A21}$$

in agreement with (4.7). The ultrarelativistic limit, $v_i = 1$, gives the extremes in (A21) and so all charges of the same sign will produce a physical null zone (single points in $\cos\theta$). The nonrelativistic limit, $v_i \to 0$, of (A18) yields

$$Q = \frac{v_1}{v_2} \quad , \tag{A22}$$

which in this limit is equivalent to

$$\frac{Q_1}{m_1} = \frac{Q_2}{m_2} \quad , \tag{A23}$$

as we expected. [Note that the third particle is not required to be nonrelativistic and thus Q_3/m_3 is not necessarily equal to the ratios in (A23); but $Q_3/p_3 \cdot q$ is equal to the ratios in (4.16).] In lowest order, (4.17) places no restriction on $\cos\theta$, consistent with the total destructive interference of dipole radiation in the nonrelativistic limit, whereas (A15) and (A22) give the first-order correction to the null zone condition, which is satisfied by $\cos\theta = 0$.

A physical null zone is guaranteed for all energies by (A23), since this condition combines with (A15) to yield

$$\cos\theta = \frac{1}{m_1 + m_2}\left(\frac{m_2}{v_1} - \frac{m_1}{v_2}\right) \quad . \tag{A24}$$

It follows from the c.m. relation $\gamma_1 m_1 v_1 = \gamma_2 m_2 v_2$, and inequalities such as $\gamma_1(1 - v_1) \leq \gamma_2(1 + v_2)$, that $|\cos\theta| \leq 1$, where $\gamma_i \equiv (1 - v_i^2)^{-\frac{1}{2}}$.

3. An $n = 4$ example

The charges and masses are taken to be the same, as in electron scattering, (4.6). This leads to a photon c.m. direction perpendicular to the beams (Fig. 2),

$$\theta = \pi/2 \quad . \tag{A25}$$

The other null zone equation (4.19) reduces to $y = x$ or

$$E_3 = E_4 \equiv E' \quad . \tag{A26}$$

Momentum conservation obviously demands that

$$\theta_3 = \theta_4 \equiv \theta' \quad , \tag{A27}$$

and the photon energy is given by

$$\omega = E - 2E' = -2E'v'\cos\theta' \quad , \tag{A28}$$

where $v_1 = v_2 \equiv v'$. It is noteworthy that the third null zone equation also leads to (A28). This is expected since the three equations, $p_1 \cdot q = p_2 \cdot q$, $p_3 \cdot q = p_4 \cdot q$, and $p_1 \cdot q = p_3 \cdot q$, are related by four-momentum conservation.

4. Null zone theorems

Let us first prove the physical null zone theorem of Sec. IV.E for the decay, $1 \to n-1+\gamma$, in the rest frame of the parent particle, mass m_1 . The $n-2$ null zone equations may be chosen to be

$$\frac{Q_i}{p_i \cdot q} = \frac{Q_2}{p_2 \cdot q} \quad , \quad i = 3, \ldots, n-1 \quad , \tag{A29}$$

with $Q_1/p_1 \cdot q$ the dependent factor, necessarily identical to the rest by (A29).

Dividing out the photon energy, we may rewrite (A29) as

$$\frac{Q_i}{m_i} \frac{1}{\gamma_i(1 - v_i \cos\theta_i)} = \frac{Q_2}{m_2} \frac{1}{\gamma_2(1 - v_2 \cos\theta_2)} \quad , \tag{A30}$$

in terms of particle speeds v_i and angles θ_i (relative to the photon). In this instance, we are given that all Q_i/m_i are equal for $i \geq 2$. Therefore, if the particles travel together, opposite to the photon ($\theta_i = \pi$, $v_i = v_2$, all i), (A30) is satisfied. This corresponds to the maximum energy for the photon and resembles the two-body decay, $m_1 \to M + \gamma$, where $M = \sum_2^{n-1} m_i \leq m_1$. Generally, (A30) is satisfied by some neighborhood phase space, as well, but we have already proven that the physical null zone is not empty, without resorting to zero photon energy.[52]

Our next consideration is the reaction $1 + 2 \to n-2+\gamma$ in the c.m. frame. One null zone equation is taken as $Q_1/p_1 \cdot q = Q_2/p_2 \cdot q$, which can always be satisfied, for $Q_1/m_1 = Q_2/m_2$, at some physical photon angle [cf. (A24)]. The remaining $n-3$ equations can be satisfied, as in decay, in the configuration where the $n-2$ final particles travel together opposite to the (fixed) photon direction.

Finally, when we have k particles in the initial state, they can be arbitrarily separated into two bunches with equal and opposite 3-momenta (c.m. frame), choosing the initial phase space region where each particle in a given bunch has the same velocity (same rest frame). These two composites have the same Q/m ratio by virtue of the identity (4.2). Thus $k-2$ equations are satisfied within the bunches, arguing as in the decay case, and another equation is satisfied for some photon angle, as in (A24). The final particles may be again clumped together opposite to the photon, satisfying another $n-k-1$ null zone equations, for a total of $n-2$. The case where the photon is in the initial state is simply the reverse of this where, as before, the identity of the initial and final state $Q/p\cdot q$ factors is guaranteed by charge and momentum conservation (the redundant null zone equation).

The physical null zone theorem for massless charges, given in Sec. IV.E, has a similar proof. (For this reason, we can consider it to be a corollary to the previous theorem.) In the general decay, $1 \rightarrow n-1+\gamma$, a physical null zone exists for the configuration where all the final state particles are massless, and travel together ($v_i = 1$) opposite to the photon. Eq. (A29) now reduces to

$$\frac{Q_i}{E_i} = \frac{Q_2}{E_2} \quad , \tag{A31}$$

and it is only necessary that the energy $m_1/2$ be divided up according to the fraction of the total charge Q_1 that each particle carries. For more general initial states, Eq. (4.18) applies to two initial particles and, by construction, to the bunched initial states for $k > 2$.

In a null zone, neutral particles must be massless and travel along with the photon (cf. Sec. VII). As such, they are easily incorporated into the physical null zone theorem and its corollary. Arbitrary numbers of them can be considered together with the photon, as a composite parallel system, and the composite energy may be partitioned in any way. [It is intriguing that all known neutral structureless (elementary) particles have mass measurements consistent with zero.]

5. General equations and remarks

To prove the physical null zone theorem, we have only needed to show that a physical point exists where the null zone condition is satisfied. We have not needed to find all such points. The determination of the complete extent of the physical null zone is increasingly complicated, particularly for more general mass and charge values, as evidenced by the earlier $n = 3$ decay analysis. Nevertheless,we can outline below an analytical approach that may be useful in the determination of physical null zones for more particles (larger n).

The $n-2$ constraints (4.1) are to be superimposed on phase space. For general decay, $1 \to n-1+\gamma$, the $3n-7$ final state variables imply a null zone with $2n-5$ dimensions. For two-body collisions, $1+2 \to n-2+\gamma$, the $3n-8$ variables imply a null zone with $2n-6$ dimensions. ($n = 3$ corresponds to a single point.) A given k-particle initial state, <u>with no symmetry axis</u>, corresponds to $3(n-k)-1$ final variables and $2n-3k+1$ null zone dimensions.

We now discuss an inductive analysis where we build larger-n null zones from smaller-n results by systematically replacing a particle by a composite of particles. For definiteness, consider the replacement of particle 3, in the $n = 3$ decay, by a composite of $n - 2$ particles. Denoting composite variables by the subscript c,

$$Q_c = \sum_3^n Q_i = Q_1 - Q_2 \quad ,$$
$$p_c = \sum_3^n p_i = p_1 - p_2 - q \quad , \tag{A32}$$

we may replace one of the $n - 2$ null zone equations by

$$\frac{Q_c}{p_c \cdot q} = \frac{Q_2}{p_2 \cdot q} \tag{A33}$$

via (4.2). Equations (4.9 - 14) and (A1 - 7) can be adapted to the case at hand by the change $3 \to c$ in the subscripts where

$$p_c^2 \equiv m_c^2 \equiv m_1^2 \mu_c^2 \quad . \tag{A34}$$

The lower (upper) limit of p_c^2 corresponding to the constituents traveling together (particle 2 at rest with zero photon energy),

$$\left(\sum_3^n m_i\right)^2 \leq p_c^2 \leq (m_1 - m_2)^2 \quad , \tag{A35}$$

but for a fixed x and y these limits are changed. The limits on x, y, and $Q \equiv Q_2/Q_c$ are found by the substitution $3 \to c$ in Eqs. (4.13), (4.14), and (A7) with the understanding that p_c^2 is evaluated at its minimum in (A35). The original discussion can be repeated here, but it must be kept in mind that the other null zone equations are not yet satisfied.

We may regard c as a two-body system made up of particle 3 and another composite d with momentum p_d and charge Q_d. To (A33) we add

$$\frac{Q_d}{p_d \cdot q} = \frac{Q_3}{p_3 \cdot q} \quad . \tag{A36}$$

This procedure can be continued, peeling away constituents from the composite and adding the null zone restrictions. At the second stage of telescoping we are led to define variables analogous to (4.10),

$$\begin{aligned} x' &\equiv \frac{2p_d \cdot q}{\Delta^2} = 1 - \frac{2E_3'}{\Delta} + \frac{m_3^2}{\Delta^2} - \frac{p_d^2}{\Delta^2} \quad , \\ y' &\equiv \frac{2p_d \cdot q}{\Delta^2} = 1 - \frac{2E_d'}{\Delta} + \frac{p_d^2}{\Delta^2} - \frac{m_3^2}{\Delta^2} \quad , \end{aligned} \tag{A37}$$

where the prime refers to the frame, $\vec{p}_1 - \vec{p}_2 = 0$, and where $\Delta^2 \equiv (p_1 - p_2)^2 = (E_1' - E_2')$.

Let us regard p_d^2 as fixed for the moment. We can show that x, y, x', y', and θ', the angle between $\vec{p}_2$ and $\vec{p}_3$ in the primed frame, represent five independent variables in terms of which all of the dot products among $p_1 \cdot p_2, p_3, p_d$, and q can be expressed. [For example $\Delta^2 = m_1^2 x/(x' + y')$ and $p_c^2 = m_1^2\, x\, (1 - x' - y')/(x' + y')$.] The point is that (A36) is easily implemented,

$$y' = \frac{Q_3}{Q_d} x' \quad . \tag{A38}$$

The next stage is to regard d as made up of particle 4 and another composite e, and so on. There remains the task of determining the nested sequence of limits on the independent variables.

An alternative procedure for smaller n or for the selection of points in the null zone, if not the whole null zone, is to rewrite the conditions

(4.1) in c.m. coordinates:

$$E_i(1 - v_i \cos\theta_i) = \frac{Q_i}{Q} E \quad , \tag{A39}$$

For fractional energy [cf. (4.10)] and fractional charge,

$$e_i \equiv 2E_i/E \quad , \tag{A40}$$

$$q_i \equiv Q_i/Q \quad , \tag{A41}$$

the relativistic version of (A39) is

$$e_i \sin^2(\theta_i/2) = q_i \quad . \tag{A41}$$

We observe, from either (A39) or (A41), that smaller charges must have less energy and/or get closer to the photon. It is essentially these equations and their implications that were used in the proof of the null zone theorem.

Finally we note that the form of the massless result (4.18) for $1+2 \to 3+\gamma$ suggests a geometrical construction where the direction of the photon at the zero can be determined and where a simple picture emerges for the zeros in reactions (1.11) and (1.13). In (4.18) let the direction of the final particle 3 lie along the hypotenuse of a right triangle with one of the sides directed along particle 2. The lengths of the hypotenuse and this side are given (weighted) by the (algebraic) charges moving in the defined directions, Q_3 and $Q_2 - Q_1$, respectively. The other side has length twice the geometric mean, $2\sqrt{Q_1 Q_2}$, of the initial charges. Thus the angle between particles 2 and 3 is θ , (4.18).

REFERENCES

1. S.J. Brodsky and R.W. Brown, Phys. Rev. Lett. 49, 966 (1982).
2. We have been informed by Mark Samuel that he has independently derived the null zone condition for the interactions of scalar particles at a point. See also M.A. Samuel, Oklahoma State University Research Note 133. (1982).
3. R.W. Brown, D. Sahdev, and K.O. Mikaelian, Phys. Rev. D 20, 1164 (1979). For a recent review, see R.W. Brown, in Proton-Antiproton Collider Physics - 1981, edited by V. Barger, D. Cline, and F. Halzen (AIP, New York, 1982), p. 251.
4. K.O. Mikaelian, M.A. Samuel, and D. Sahdev, Phys. Rev. Lett. 43, 746 (1979). The zeros and factorization of the cross sections for (1.11) and (1.13) can also be seen in the curves and formula of Ref. 3. An earlier reference for this factorization is K.O. Mikaelian, Phys. Rev. D17, 750 (1978).
5. C.J. Goebel, F. Halzen, and J.P. Leveille, Phys. Rev. D 23, 2682 (1981).
6. Z. Dongpei, Phys. Rev. D 22, 2266 (1980). See also K.O. Mikaelian, Phys. Rev. D 25, 66 (1982).
7. T.R. Grose and K.O. Mikaelian, Phys. Rev. D 23, 123 (1981).
8. As an alternative to using δ_i, all particles could be defined as outgoing, with the replacement $Q \to -Q$ and $p \to -p$ to be made for incoming particles. Note that $Q/p\cdot q$ is invariant under this replacement.
9. L.D. Landau and E.M. Lifshitz, The Classical Theory of Fields (Pergamon Press, Oxford, 1975), Ch. 9.
10. J.D. Jackson, Classical Electrodynamics (Wiley, New York, 1975), Chs. 14 and 15. See problems 15.3 and 15.5 of this reference.
11. The magnetic dipole radiation depends upon the second (first) time-derivative of the magnetic moment if one is dealing with orbital (intrinsic) magnetic moments (cf. Ref. 9, p. 189 and Ref. 10, p. 672).

12. D.R. Yennie, Lectures on Strong and Electromagnetic Interactions (Brandeis, Massachusetts, 1963), p. 165.

13. Where needed, the complex conjugation of polarization vectors is left understood.

14. Various names for the $Q/p.q$ factor might be given based on the relation of the denominator $p.q$ to retarded time (see Sec. IX), to Doppler shifts, or to light-cone variables. In a frame where $\vec{q} = \omega\hat{z}$, the light-cone coordinates for the photon are $q^+ = q^o + q^z = 2\omega$ and $q^- = q^o - q^z = 0$. For particles with mass, we can use x and $k_\perp^2$ such that $p^+ = x\,q^+$ and $p^- = (m^2 + k_\perp^2)/p^+$. Therefore, $Q/p.q = Q/(m_\perp^2/x)$.

15. The restrictions studied in this paper, such as the positivity condition in (4.7) and the conditions for neutral particles discussed in Sec. VII, strongly limit the occurrence of radiation zeros. For example, the zero in $u\bar{d} \to W^+\gamma$ of (1.11) exists in the physical region only because both charges of the fermion and the antifermion have the same sign. The fact that radiation zeros were not discovered long ago may be attributed to these stringent conditions.

16. S. Weinberg and G. Feinberg, Phys. Rev. Lett. 3, 111 (1959).

17. For particles with spin the Ward-Takahashi identity does not suffice to determine $\varepsilon\cdot\Gamma$ in terms of the appropriate full propagators $\hat{\Delta}$. In general [cf. A. Salam, Phys. Rev. 130, 1287 (1963)], $\Gamma^\mu = \Gamma_A^\mu + \Gamma_B^\mu$, where $\Gamma_A^\mu = (p'+p)^\mu(p'^2-p^2)^{-1}[\hat{\Delta}(p'^2)^{-1} - \hat{\Delta}(p^2)^{-1}]$ and $(p'-p)\cdot\Gamma_B = 0$. However, $\varepsilon\cdot\Gamma_B \neq 0$, although $\varepsilon\cdot\Gamma_A$ clearly satisfies a spin-indexed version of (5.3). For this reason Γ_A^μ might be referred to as the convective part of Γ^μ.

18. J.D. Bjorken and S.D. Drell, Relativistic Quantum Mechanics (McGraw-Hill, New York, 1964), Ch. 2.

19. Equivalently, we may speak of $\Delta\Gamma_i$ as proportional to the first-order change in Γ_i, viewed as a Lorentz tensor in the spinor representation. As examples, $\Delta\Gamma_i = 0, 0, \omega_{\mu\nu}\gamma^{\nu}, \omega_{\mu\nu}\gamma^{\nu}\gamma_5, \omega_{\mu\nu}\omega_{\tau\rho}\sigma^{\nu\rho}$ for $\Gamma_i = 1, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5, \sigma_{\mu\tau}$, respectively. Even though the outer product $\Gamma_1 \cdots \Gamma_D$ in (5.8) is not tied together in spin space, it must be an overall rank-zero Lorentz tensor $S^{-1}\Gamma_1 SS^{-1}\Gamma_2 S \cdots S^{-1}\Gamma_D S = \Gamma_1\Gamma_2\cdots\Gamma_D$ or, in first order, $\sum_i^D \Delta\Gamma_i \prod_{j\neq i}^D \Gamma_j = 0$.

20. V_G is a multi-spinor-indexed matrix, $(V_G)_{\alpha\beta\cdots}$, in general, with an index for each internal leg and where each index may be regarded as an internal spinor wave function. For example, $(V_G)_\alpha = (V_G)_\delta w^\delta(\alpha)$ for $w^\delta(\alpha) = g_\alpha^\delta$. In this way we may say that each wave function of the vertex v, external or internal, is transformed with the same $\omega_{\mu\nu}$ by the photon emission associated with the corresponding vertex leg. This also clarifies the description of V_G and $M_\gamma(V_G)$ as rank-zero Lorentz tensors and should be compared to the case where vector particles are included and can lead to a higher rank for these tensors.

21. Ref. 18, Ch. 10.

22. The Feynman rules are listed, for example, in Refs. 3 and 6. For comparison, note that $Q = -e < 0$ for the quanta of the W field, the W^- particles, in Ref. 3.

23. The reader is warned that, in the separation of terms in (5.37) leading to the contact current in (5.39), the residual convection current has the coefficient $p\cdot r$ and <u>not</u> $(p\pm q)\cdot r$. The momentum shift is included in (5.39).

24. They also arise for couplings with higher-derivatives, irrespective of spin.

25. No bi-difference sum in $(a_i - a_j)(b_i - b_k)$ exists for both $\sum_i a_i = \sum_i b_i = 0$, since it is now impossible to have either identical a_i or identical b_i. However, bi-difference forms can be constructed via a multiplier λ_i, e.g., such that $\sum_i (a_i - a_j)\lambda_i = 0$. Then by Lemma 1, $s = \sum_i (a_i - a_j)\lambda_i(b_i/\lambda_i - b_k/\lambda_k)$. The bi-difference expansion in Lemma 2 can be obtained in this way noting that identical A_i/C_i, identical B_i/C_i, and Eq. (6.3) can all coexist.

26. We borrow notation from Ref. 5 in developing what is essentially a generalization of their factorization formula.

27. A naive generalization of (6.5b) would be

$$\sum_i^{\ell} A_i B_i / C_i = C_k^{-1} \sum_{i<j} C_i C_j (A_i/C_i - A_j/C_j)(B_j/C_j - B_i/C_i) \quad \text{for} \quad i,j \neq k ,$$

but has $(\ell-1)(\ell-2)/2$ terms. The minimal form is (6.4).

28. The linear relationship is $\sum_i^{n_v} \Delta_{ij}(Q \text{ or } \delta J)\delta_i p_i \cdot q = 0$.

29. A neutral particle is one with no photon couplings. Particles with zero charge but nonzero magnetic moments are non-gauge-theoretic and never satisfy the conditions of the interference theorem.

30. We are guaranteed the presence of an amplitude zero if the null zone condition is satisfied first and then the $Q_r \to 0$ limit is taken. We are concerned in this section with the reverse order, $Q_r = 0$, *ab initio*, the physically relevant case.

31. Of course, it may be that (7.5) conflicts with the limits on the physical region. The general physical null zone discussion in the Appendix is easily adapted to incorporate a massless neutral particle. (A neutral particle with mass can be included to the extent that it is very relativistic and its mass is negligible.) The constraints on the Q_i/m_i ratios in the physical null zone theorem need only apply to the remaining charged particles.

32. Recall that the massless limit of a vector particle is generally singular and involves the problem of eliminating the third spin state. A mechanism for the elimination of the extraneous helicity component is a conserved current.

33. As another example, there is no radiation zero in Fig. 9 when particle 1 has no charge. Although it is possible to construct a conserved current by considering particle 3 to be external and to have the same mass as particle 2, the null zone is in the forward direction.

34. A classic paper on mass singularities is T. Kinoshita, J. Math. Phys. 3, 650 (1962).

35. See the remarks by K.A. Johnson, MIT unpublished report, CTP 977, March 1982.

36. G. 't Hooft and M. Veltman, Diagrammar, CERN unpublished report, 73-9 (1973).

37. The limit $q \to 0$ implied by $O(q)$ can be taken within the physical region, although it requires the existence of a physical threshold in a two-body reaction. [See (1.11).] Thus what we mean by $O(q)$ is consistent with momentum conservation at any vertex. In the case of additional massless particles, whose thresholds lie at $q = 0$, we can have nonanalytic behavior, such as $O(q \ln q)$, for example, in place of $O(q)$ in (8.1). This infrared problem is left understood in the standard low-energy theorem statement.

38. F.E. Low, Phys. Rev. 110, 974 (1958).

39. S.L. Adler and Y. Dothan, Phys. Rev. 151, 1267 (1966).

40. A closed-loop graph can be expanded in external momenta, leading to an effective derivative-coupling series. In this way, conclusions found for derivative couplings (such as the need for seagulls) can be applied to closed loops.

41. S.D. Drell and A.C. Hearn, Phys. Rev. Lett. 16, 908 (1966); S.B. Gerasimov, J. Nucl. Phys. (USSR) 2, 598 (1965) [translation: Sov. J. Nucl. Phys. 2, 430 (1966)].

42. See, for example, T.-P. Cheng and L.-F. Li, Phys. Rev. Lett. 38, 381 (1977).

43. S. Gasiorowicz, Elementary Particle Physics (Wiley, New York, (1966), Ch. 4. We thank W. Bardeen and G. Mack for discussions about the little group.

44. We therefore have an explanation for the "spatial generalized Jacobi identity" in Eq.(14) of Ref. 5. (A factor of 2 $p\cdot p_1$ is missing.) Only the single instance of the quadratic Yang-Mills term, implicit in their identity and discussed in Sec. V, goes beyond the linear Poincare invariance argument. Also, from our general arguments we see that additional $n = 3$ vertices beyond the class of interactions considered in Ref. 5 qualify for factorization (the radiation interference theorem in the $n = 3$ case).

45. See, for example, R.J. Hughes, Phys. Lett. 97B, 246 (1980) and Nucl. Phys. B186, 376 (1981).

46. Notice that the spin currents in Table I can be written in terms of the finite transformation (9.4). Each spin current is proportional to $S^{\mu\nu}\omega_{\mu\nu}$ which is in turn proportional to $S^{\mu\nu}\Lambda_{\mu\nu}$. Furthermore, the Dirac transformation (5.15) is correct for finite λ, according to (9.4).

47. Ref. 10, Ch. 11.

48. V. Bargmann, L. Michel and V.L. Telegdi, Phys. Rev. Lett. 2, 435 (1959); S.J. Brodsky and J.R. Primack, Ann. Phys. 52, 315 (1969).

49. Ref. 10, Ch. 14.

50. See Mikaelian, Ref. 6, for virtual photon effects in the $n = 3$ case.

51. The vanishing of the q^0 and q^1 terms is due to Poincaré invariance and can be considered as a cancellation in flat-space. The q^2 Yang-Mills cancellation is at the basis of gauge theory and can be interpreted to be a curved-space symmetry.

52. A more general study of the limits on the physical null zone theorem and of $n = 4$ has now been made (S.G. Naculich, G. Passarino, private communications).

53. M. Hellmund and G. Ranft, Z, Phys. C12, 333 (1982). See also K.J.F. Gaemers and G.J. Gounaris, Z. Phys. C1, 259 (1979).

54. M.L. Laursen, M.A. Samuel, A. Sen, and G. Tupper, Oklahoma State University Research Note 137 (1982). The zeros found in this study that persist to all orders are due to angular momentum constraints.

55. C.H. Llewellyn-Smith, Phys. Lett. 46B, 233 (1973); J.M. Cornwall, D.N. Levin, and G. Tiktopoulos, Phys. Rev. Lett. 30, 1268; 31, 572(E) (1973).

56. C.J. Goebel (private communication).

57. D. Cline and C. Rubbia (private communication).

58. This has been suggested by R. Decker (private communication).

59. E.A. Paschos (private communication).

60. R. Gastmans, Lectures in 18th Karpacz Winter School of Theoretical Physics, Poland (1981).

TABLE CAPTION

Table I. The rules for the construction of the radiation vertex expansion, (5.22), for radiation amplitudes generated by any source tree graph with gauge-theoretic couplings. The factors modify the external or internal leg of each source vertex and are derived in Sec. V. All propagators are included in the factors R in (5.22). There is no momentum shift from derivative couplings in the coefficient of the convection current since this product is included in the contact current (see Ref. 23). In the Yang-Mills vertex, however, the coefficient of the spin currents includes the momentum shift, yielding the quadratic terms discussed in Sec. V.E. The radiation decomposition identity is also shown (generalized to include possible contact currents) from which internal-leg factors are derived.

TABLE I

RADIATOR	FACTOR	POSITION
vertex leg with charge Q along momentum p (or p + q) before emitting photon with momentum q and polarization ε, seagull included (if any)	$\frac{Q}{p\cdot q} j$	factor goes between wave function and vertex in source graph (internal wave functions are Kronecker δ-functions in spin space)

Current $\boxed{j = j_{conv} + j_{spin} + j_{cont}}$

where

j_{conv} = (first-order coefficient in) universal displacement of wave function = $\pm\ p\cdot\varepsilon$ for outgoing (+) or incoming (−),

j_{spin} = (first-order coefficient in) universal Lorentz transformation of wave function = $\{0;\ +\frac{i}{4}\bar{\sigma}^{\alpha\beta}\omega_{\alpha\beta};\ -\frac{i}{4}\sigma^{\alpha\beta}\omega_{\alpha\beta};\ g_{\alpha\beta}\rightarrow\omega_{\alpha\beta}\}$ for {scalar ; spinor $\bar{u},\bar{v}$; spinor u,v ; vector $\eta_\alpha = g_{\alpha\beta}\eta^\beta, \eta^+_\alpha = g_{\alpha\beta}\eta^{+\beta}$} ,

j_{cont} = (first-order coefficient in) universal Lorentz transformation of derivative coupling, $g_{\alpha\beta} \rightarrow \omega_{\alpha\beta}$ for $p_\alpha = g_{\alpha\beta}p^\beta$,

with

$$\omega_{\alpha\beta} = q_\alpha\varepsilon_\beta - \varepsilon_\alpha q_\beta \quad .$$

Decomposition Identity $\boxed{D(p-q)\Gamma D(p) + \text{seagulls (if any)} = D(p-q)j\,\frac{Q}{p\cdot q} + \frac{Q}{p\cdot q}j\,D(p)}$

where

	scalar	Dirac	anti-Dirac	vector
propagator D(p)	$i(p^2 - m^2)^{-1}$	$i(\not{p} - m)^{-1}$	$i(-\not{p} - m)^{-1}$	$iP_{\mu\nu}(p)(p^2 - m^2)^{-1}$ Eq. (5.27)
photon vertex $\Gamma(p-q,q,p)$	$-iQ(2p-q)\cdot\varepsilon$	$-iQ\not{\varepsilon}$	$+iQ\not{\varepsilon}$	$iQ\gamma_{\alpha\beta\gamma}(p-q,q,-p)\varepsilon^\beta$ Fig.7

FIGURE CAPTIONS

Fig. 1. a) The general amplitude for photon emission in the interactions of n particles, $k \to n - k + \gamma$. b) A contribution with an infrared divergence.

Fig. 2. The amplitude zero in $e^-e^- \to e^-e^-\gamma$ occurs when both the photon is at right angles to the c.m. beams and the final electrons have equal energies. This is a two-dimensional null zone: E',ϕ' or θ',ϕ' at fixed $\theta = \pi/2$.

Fig. 3. a) The n-vertex source graph. b) A photon attachment to an external leg.

Fig. 4. a) A sample tree source graph and b) its associated radiation amplitude, as defined in Sec. II.

Fig. 5. The radiation decomposition identity for the coupling of an external photon to an internal particle line. A double line represents a propagator. A dashed line is quasi-external in that the calculation of each current on the right-hand-side is carried out as if the dashed line were real. See Eqs. (5.3), (5.21), and (5.31). Additional contributions to the left-hand-side due to seagull graphs where the photon is attached to either end are easily incorporated into the respective quasi-external factor on the right-hand-side. See Table I.

Fig. 6. The radiation amplitude for an $n = 4$ tree source graph.

Fig. 7. The Feynman rule for a Yang-Mills locally gauge-invariant three-vertex for vector fields, with four-momenta a,b,c and polarization indices α,β,γ . The coupling constant g would be augmented by a matrix representation for the general internal-symmetry gauge group. In the U(1) case where a vector boson with charge Q emits a photon, we have $g = Q$. See Ref. 22.

Fig. 8. An example of photon-emission by an incoming or outgoing particle, with momentum p and charge Q, that is coupled through a derivative ∂_β of its own field to other particles. The seagull factor is $-Q\,g_{\beta\mu}$ for photon polarization ε^μ.

Fig. 9. The source graph example of Sec. V.E. The vector, Dirac, and scalar particles are denoted by V,D, and S, respectively. The bottom vertex includes a scalar fermion current.

Fig. 10. The amplitude for radiative decay, $1 \rightarrow 2 + 3 + \gamma$, separated into (a) radiation from the external legs and (b) internal radiation including seagulls.

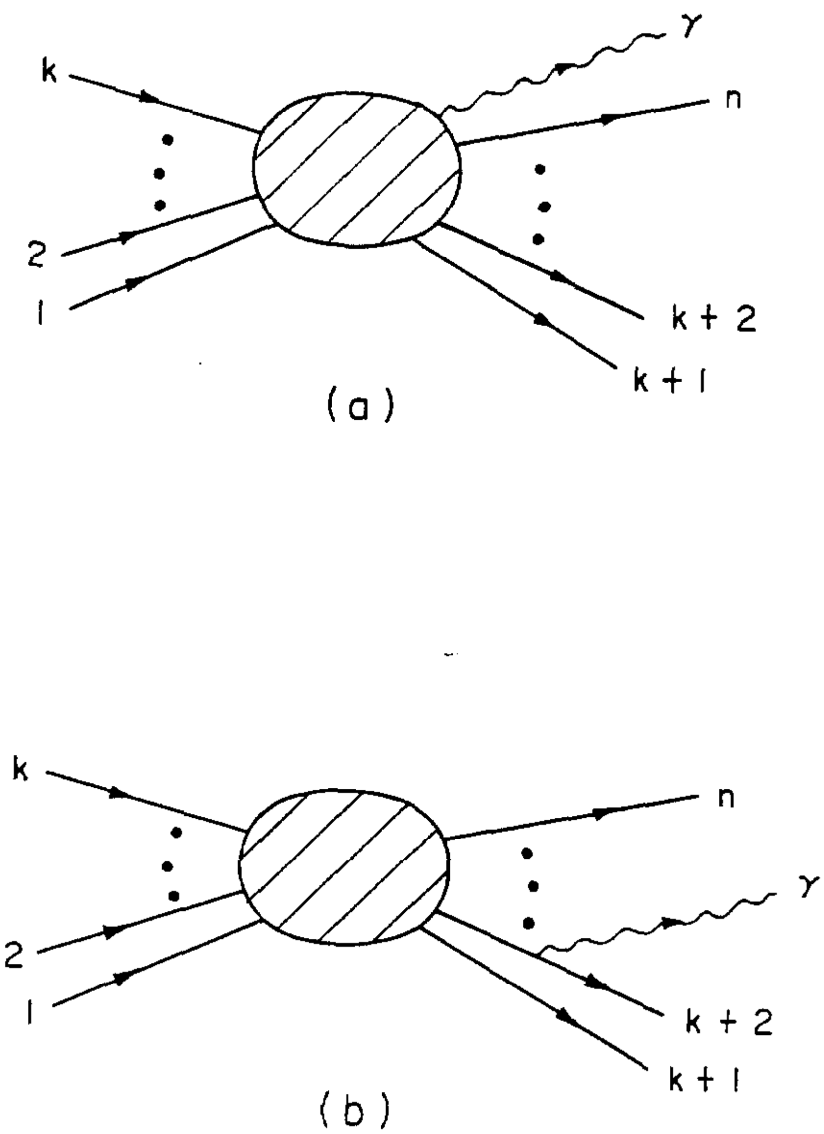


Fig. 1

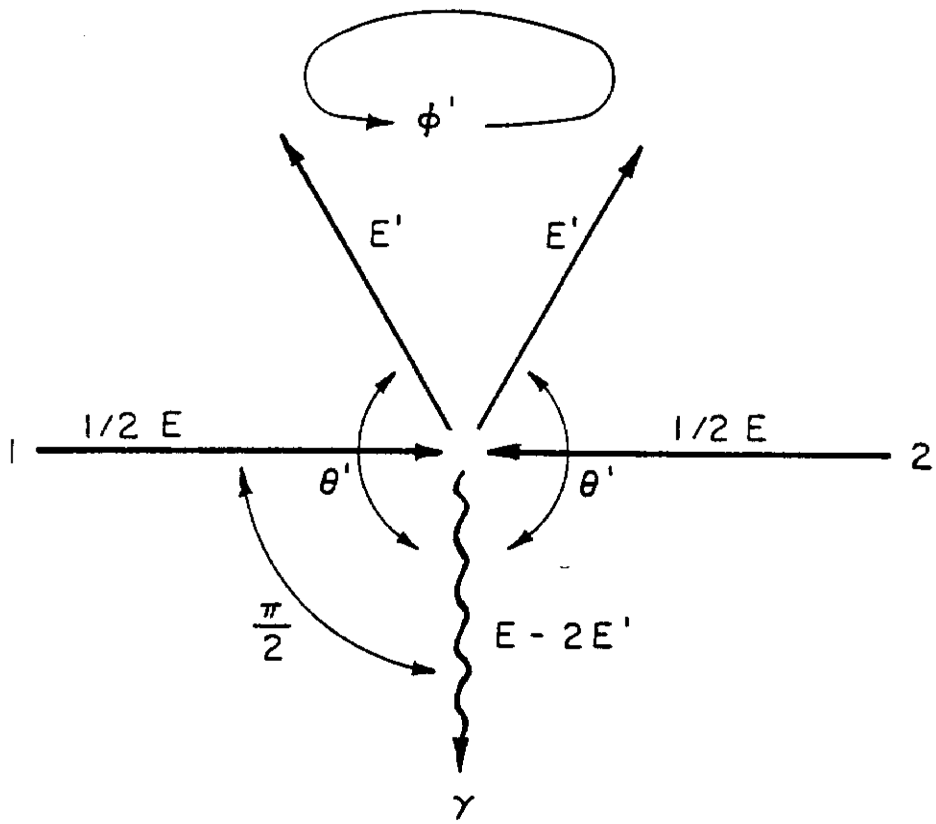


Fig. 2

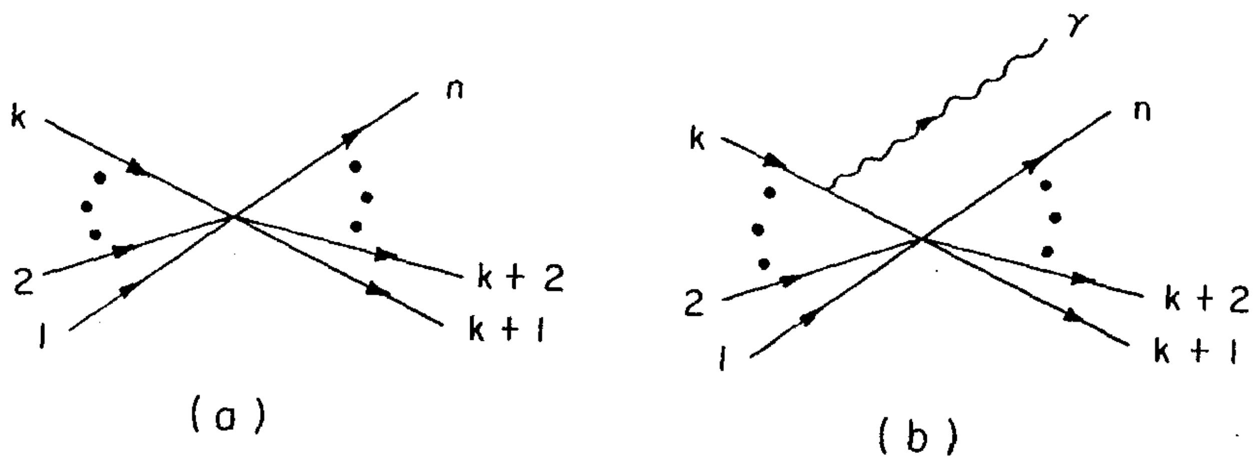
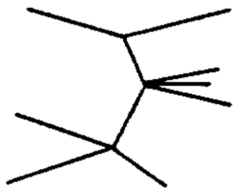
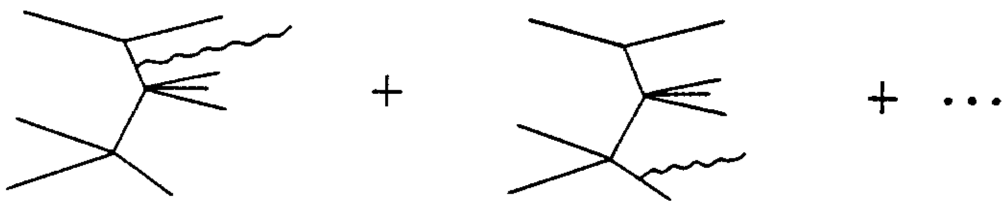


Fig. 3



(a)



(b)

Fig. 4

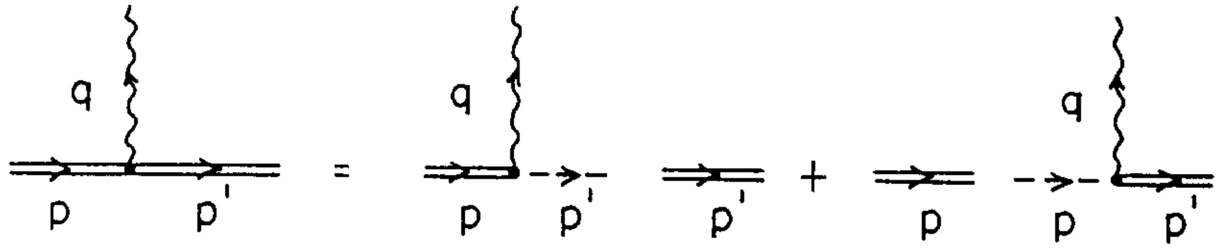


Fig. 5

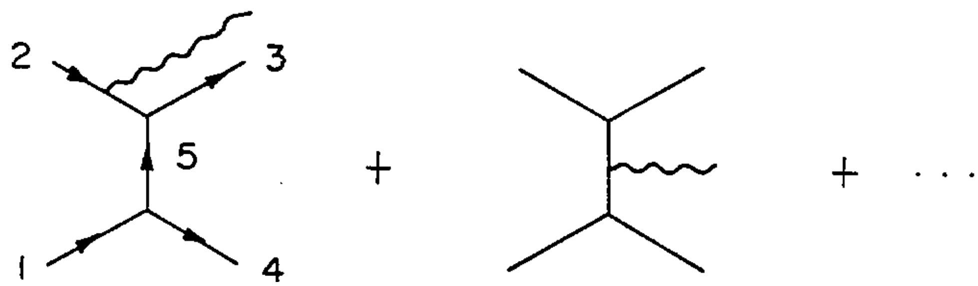
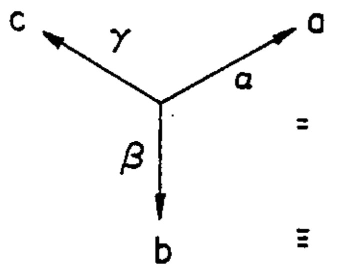


Fig. 6



$$= ig\left[g_{\alpha\beta}(b-a)_\gamma + g_{\beta\gamma}(c-b)_\alpha + g_{\gamma\alpha}(a-c)_\beta\right]$$

$$\equiv ig\ Y_{\alpha\beta\gamma}(a,b,c)$$

$$a+b+c = 0$$

Fig. 7

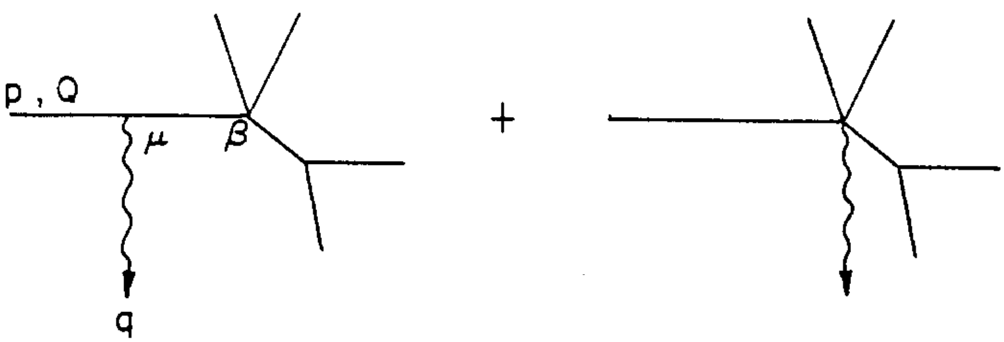


Fig. 8

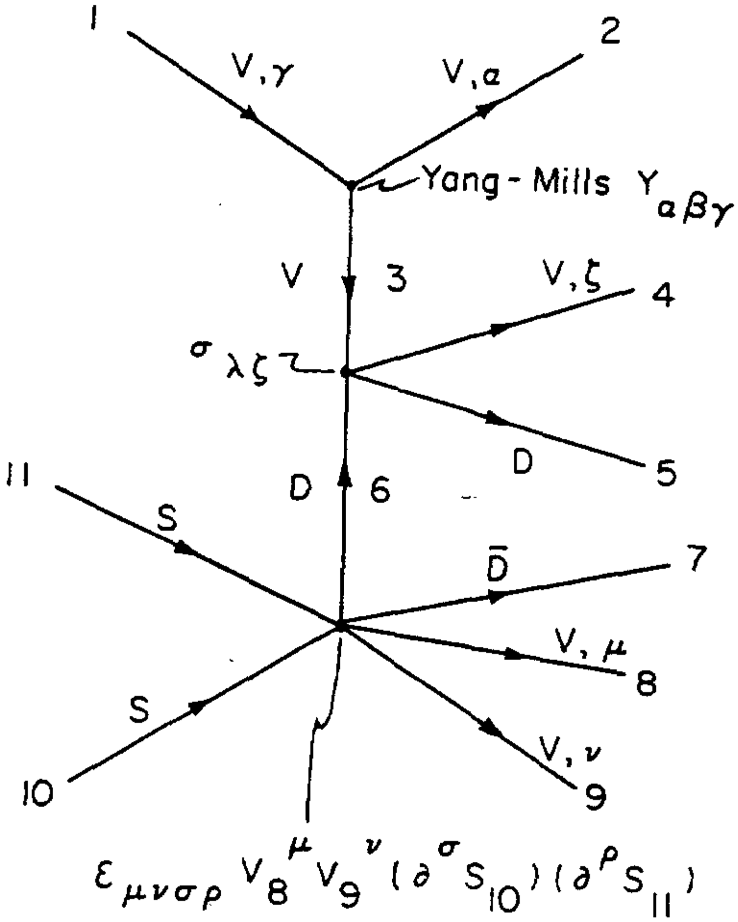

1
V, γ
V, α
2
Yang - Mills Y αβγ
V
3
V, ζ
4
σ λζ
D
5
11
S
D
6
D̄
7
V, μ
8
S
V, ν
10
9
ε μνσρ V8^μ V9^ν (∂^σ S10) (∂^ρ S11)

Fig. 9

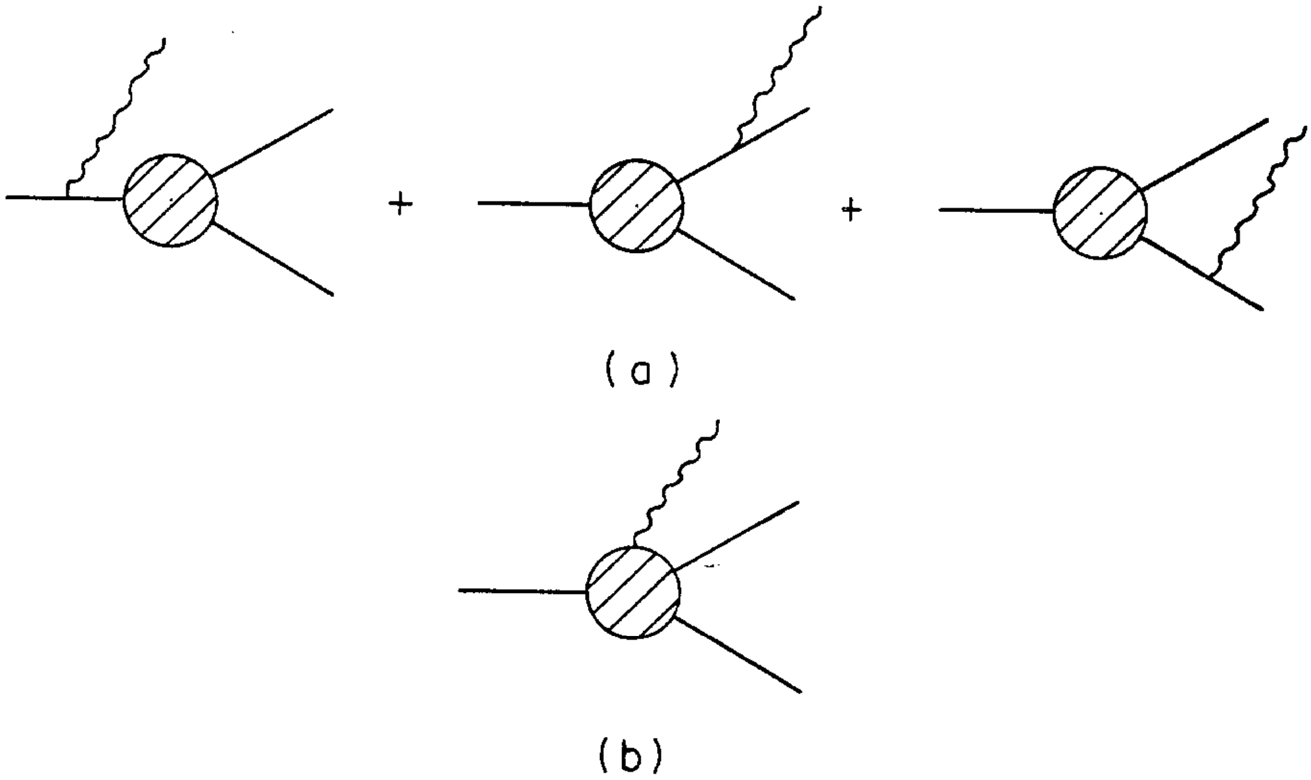


Fig. 10