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ANGULAR AND PHASE ACCEPTANCE OF A QUADRUPOLE BEAM TRANSPORT CHANNEL

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I. Introduction

The possibility of computing the angular and phase acceptance of a long beam transport channel will be of importance in the 200 GeV accelerator project where a 5 km transport line is considered between the extraction point and the boundary site.[1]

We attempt to derive here general and simple formulae for computing the angular and phase acceptances of a multiplet, composed of an arbitrary number of arbitrary quadrupoles. Applications are then made to particular structures.

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II. Notations

Consider a multiplet comprising n AG quadrupoles. Let

$$Q_{c,j} = \begin{Vmatrix} \cos\theta_j & \frac{1}{k_i}\sin\theta_j \\ -k_j\sin\theta_j & \cos\theta_j \end{Vmatrix} \tag{1}$$

be the transfer matrix of the j^{th} quadrupole in its focusing plane and

$$Q_{d,j} = \begin{Vmatrix} ch\theta_j & \frac{1}{k_j} sh\theta_j \\ k_j sh\theta_j & ch\theta_j \end{Vmatrix} \tag{2}$$

the corresponding transfer matrix for the defocusing plane.

In these relations we have put

$$k_j^2 = \frac{G_j}{P/Q} \tag{3}$$

$$\theta_j = k_j s_j, \tag{4}$$

G_j being the magnetic gradient in the j^{th} quadrupole of the array, P the particle momentum, Q its charge and s_j the length of the j^{th} quadrupole. We assume in all cases $\theta_j < \frac{\pi}{2}$. The subscripts 1, 2, 3, ... j, ... n will be used to specify the quadrupoles in the direction of the propagation of the beam. The drift lengths will be denoted by L with the corresponding subscripts and the useful aperture of the quadrupoles will be denoted by R; we assume that R has the same value for all quadrupoles. The object distance or distance of the source point from the entrance of the first quadrupole will be called p, the corresponding image distance q.

III. Angular Acceptance of a General Multiplet

Consider a point source placed on the axis of the multiplet and emitting a beam of half-angular divergence x'_0 in the cdcd ... plane of the multiplet (i.e. the plane where the multiplet starts with a focusing quadrupole).

The beam envelope can have a maximum only in a focusing lens. Let then j be a quadrupole where a maximum takes place and

$$M_j' = \begin{Vmatrix} A_j' & B_j' \\ C_j' & D_j' \end{Vmatrix} \qquad (5)$$

the transfer matrix of the incomplete multiplet which extends from the entrance of the multiplet to the transverse plane where the beam grazes the aperture. This situation is described by the matrix relation

$$\begin{pmatrix} R \\ 0 \end{pmatrix} = \begin{Vmatrix} A_j' & B_j' \\ C_j' & D_j' \end{Vmatrix} \times \begin{Vmatrix} 1 & p \\ o & 1 \end{Vmatrix} \begin{pmatrix} 0 \\ x_o' \end{pmatrix} \qquad (6)$$

or more explicitely by means of the two equations

$$R = \left(A_j' \, p + B_j'\right) x_o' \qquad (7)$$

and

$$0 = C_j' \, p + D_j' \qquad (8)$$

Taking into account the unimodularity condition, we can replace this system of equations by the equivalent system

$$C_j' \, p + D_j' = 0 \qquad (9)$$

$$x_o' = -RC_j' \qquad (10)$$

which involves only the two matrix elements C_j' and D_j'. Let now s_j' be the distance from the entrance of the j^{th} quadrupole to the transverse plane where the maximum excursion occurs. If we write

$$\theta_j' = k_j s_j' \qquad (11)$$

we can calculate the matrix M_j' by means of the relation

$$\begin{Vmatrix} \cos(\theta_j - \theta_j') & \frac{1}{k_j} \sin(\theta_j - \theta_j') \\ -k_j \sin(\theta_j - \theta_j') & \cos(\theta_j - \theta_j') \end{Vmatrix} \times M_j' = M_j \qquad (12)$$

where

$$M_j = \begin{Vmatrix} A_j & B_j \\ C_j & D_j \end{Vmatrix} \tag{13}$$

is the transfer matrix of the complete multiplet which starts at the entrance of the first lens and stops at the exit of the j^{th} lens (i. e. the lens where the maximum beam deviation is supposed to occur).

From Eq. (12) we find

$$C_j' = A_j k_j \sin (\theta_j - \theta_j') + C_j \cos (\theta_j - \theta_j') \tag{14}$$

$$D_j' = B_j k_j \sin (\theta_j - \theta_j') + D_j \cos (\theta_j - \theta_j') \tag{15}$$

Replacing these expressions in Eq. (9) and solving for $\theta_j - \theta_j'$, we find

$$\boxed{\cot (\theta_j - \theta_j') = - k_j \frac{A_j p + B_j}{C_j p + D_j}} \tag{16}$$

where

$$q_j = - \frac{A_j p + B_j}{C_j p + D_j} \tag{17}$$

is simply the image point produced by the first j lenses of the multiplet.

Finally, Eq. (10) gives

$$x_{oj}' = \frac{R}{\sqrt{(A_j p + B_j)^2 + \frac{1}{k_j^2} (C_j p + D_j)^2}} \tag{18}$$

Here we have written x'_{oj} instead of x'_o to indicate that it corresponds to a situation where the beam grazes the limiting aperture in the j^{th} lens.

In writing Eq. (18) we assume that the images produced by the multiplet are real, i.e.,

$$(A_n p + B_n)(C_n p + D_n) < 0 \tag{19}$$

To a positive displacement corresponds then a negative slope at the exit of the multiplet and vice versa. The stronger condition

$$(A_j p + B_j)(C_j p + D_j) < 0 \tag{20}$$

is however not necessary; it simply means that there is no maximum in the j^{th} quadrupole if Eq. (20) is not satisfied.

Eq. (18) is a convenient formula for computing acceptances because in designing the multiplet one would have to compute anyway the matrix elements A_j, B_j, C_j, D_j with j going from 1 to n. At each other step (every other lens is defocusing) one can then compute x'_{jo} from (18) and take the minimum value

$$X'_o = \mathrm{Min}\,(x'_{oj}) \qquad j = 1, 3, 5, \ldots \tag{21}$$

In the perpendicular plane where the succession of lenses follows the pattern dcdc ... the same formula holds except that we now take

$$Y'_o = \mathrm{Min}\,(y'_{oj}) \qquad j = 2, 4, 6, \ldots \tag{22}$$

The total angular acceptance of the multiplet can then be defined as

$$\Omega = 4\, X'_o Y'_o \tag{23}$$

and expressed in steradians.

IV. Radial Acceptance of a General Multiplet

We now consider a beam of radial extension $2x_o$ emitting rays parallel to the axis in the cdcd ... plane. As before, the beam envelope can have a maximum only in a focusing lens and if j is a quadrupole where the beam grazes the limiting aperture, the situation is described by

$$\begin{pmatrix} R \\ 0 \end{pmatrix} = \begin{Vmatrix} A_j' & B_j' \\ C_j' & D_j' \end{Vmatrix} \begin{pmatrix} x_o \\ 0 \end{pmatrix} \tag{24}$$

or more explicitely by

$$R = A_j' x_o \tag{25}$$

$$C_j' = 0 \tag{26}$$

From Eq. (12) we have

$$A' = A_j \cos(\theta_j - \theta_j') - \frac{C_j}{k_j} \sin(\theta_j - \theta_j') \tag{27}$$

$$C' = A_j k_j \sin(\theta_j - \theta_j') + C_j \cos(\theta_j - \theta_j') \tag{28}$$

so that Eq. (26) leads to

$$\boxed{\cot(\theta_j - \theta_j') = -\frac{A_j k_j}{C_j}} \tag{29}$$

where

$$q_j = -\frac{A_j}{C_j} \tag{30}$$

is simply the position of the image focal plane of the first j lenses of the multiplet.

On the other hand, Eq. (25) gives for the maximum accepted excursion of a ray grazing the aperture in the j^{th} quadrupole

$$\boxed{x_{oj} = \frac{R}{\sqrt{A_j^2 + \frac{1}{k_j^2} C_j^2}}} \tag{31}$$

Here again we have written x_{oj} instead of x_o to indicate that it corresponds to a situation where the beam grazes the limiting aperture in the j^{th} lens.

In fact, Eqs. (29) and (30) can be obtained respectively from Eqs. (16) and (17) by letting $p \longrightarrow \infty$ whereas Eq. (31) can be obtained from Eq. (18) by letting p go to infinity, x'_{oj} to zero, and px'_{oj} to x_{oj}.

Eq. (31) is a convenient formula for computing radial acceptances of a general multiplet. As in the case of angular acceptance, one can calculate the matrix elements A_j, B_j with j going from 1 to n and at each other step one can determine x_{oj} from Eq. (31). The minimum value

$$X_o = \text{Min}\,(x_{oj}) \qquad j = 1,\ 3,\ 5,\ \ldots \tag{32}$$

represents the radial acceptance in the cdcd ... plane. In the perpendicular plane where the succession of lenses follows the pattern dcdc ... we take

$$Y_o = \text{Min}\,(x_{oj}) \qquad j = 2,\ 4,\ 6,\ \ldots \tag{33}$$

to obtain the radial acceptance.

V. Phase Acceptance of a General Multiplet

It follows from the formulae given above that in the cdcd ... plane the phase acceptance can be defined as

$$A_x = \pi X_o X'_o \tag{34}$$

whereas in the dcdc ... plane the phase acceptance is given by

$$A_y = \pi Y_o Y'_o \tag{35}$$

The total phase volume is therefore

$$A = \pi^2 X_o Y_o X_o' Y_o' \; ; \tag{36}$$

it can be expressed in m^2 steradians.

VI. Some Properties of Invariance

In the preceding theory A_j, B_j, C_j, D_j represent the matrix elements of the multiplet at the exit of the j^{th} lens, which is supposed to be focusing. For calculating the acceptances one may however just as well use the matrix elements at the entrance of the j^{th} lens i.e. at the end of the drift space which follows the $(j-1)^{th}$ defocusing lens without having to change anything in the formulae expressing the acceptances. This is due to the following invariance property:

For any matrix

$$\begin{Vmatrix} \alpha & \beta \\ \gamma & \delta \end{Vmatrix}$$

the quantities

$$k^2\alpha^2 + \gamma^2$$

$$k^2\beta^2 + \delta^2$$

and

$$k^2\alpha\beta + \gamma\delta$$

remain invariant under the effect of a premultiplication by a focusing matrix; in other words, if

$$\begin{Vmatrix} a & b \\ c & d \end{Vmatrix} = \begin{Vmatrix} \cos\theta & \frac{1}{k}\sin\theta \\ -k\sin\theta & \cos\theta \end{Vmatrix} \times \begin{Vmatrix} \alpha & \beta \\ \gamma & \delta \end{Vmatrix} \tag{37}$$

then

$$k^2a^2 + c^2 = k^2\alpha^2 + \gamma^2 \tag{38}$$

$$k^2 b^2 + d^2 = k^2 \beta^2 + \delta^2 \tag{39}$$

$$k^2 ab + cd = k^2 \alpha\beta + \gamma\delta \tag{40}$$

These properties are readily checked by a straight forward calculation and using them in Eqs. (18) and (31) one can immediately show that for calculating the acceptances it does not matter whether one uses the matrix elements at the entrance or at the exit of the j^{th} lens. In some cases it will be simpler to use the entrance matrix, in some other cases the reverse may be true. For example, in the case of a symmetric triplet a complete cdc matrix is symmetric and shows transparent properties whereas a cd matrix followed by a drift space is a dead end, if not conveniently rearranged.

We may notice a similar invariance property which applies to defocusing conditions:

For any matrix

$$\begin{Vmatrix} \alpha & \beta \\ \gamma & \delta \end{Vmatrix}$$

the quantities

$$k^2\alpha^2 - \gamma^2$$

$$k^2\beta^2 - \delta^2$$

$$k^2\alpha\beta - \gamma\delta$$

remain invariant under the effect of a premultiplication by a defocusing matrix.

Finally for any matrix

$$\begin{Vmatrix} \alpha & \beta \\ \gamma & \delta \end{Vmatrix}$$

the quantities

$$\alpha^2 \pm k^2\beta^2$$
$$\gamma^2 \pm k^2\delta^2$$
$$\alpha\gamma \pm k^2\beta\delta$$

remain invariant under the effect of a postmultiplication by a focusing (defocusing) matrix.

We shall not have the opportunity to use the last relations, we conclude however with a remark which follows directly from the preceding considerations: if the channel acceptance is limited by the j^{th} lens, the length of this lens does not enter the acceptance formulae but only its focusing strength k_j.

VII. Application to the Doublet

To calculate the angular and radial acceptance in the cd plane, we can use, according to the general theory, the entrance elements of the matrix M_j with j = 1, i. e.

$$M_1 = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} \tag{41}$$

Putting

$$k_1 p = P_1 \tag{42}$$

we then have from Eq. (18)

$$X_o' = \frac{k_1 R}{\sqrt{1 + P_1^2}} \tag{43}$$

whereas Eq. (31) yields, as expected,

$$X_o = R \tag{44}$$

In the cd plane the phase acceptance is therefore

$$A_x = \frac{\pi k_1 R^2}{\sqrt{1 + P_1^2}} \tag{45}$$

To calculate the acceptances in the dc plane, we use the entrance matrix M_j with $j = 2$, i.e.

$$M_2 = \begin{Vmatrix} 1 & L \\ 0 & 1 \end{Vmatrix} \times \begin{Vmatrix} \mathrm{ch}\,\theta_1 & \frac{1}{k_1}\,\mathrm{sh}\,\theta_1 \\ k_1 \mathrm{sh}\,\theta_1 & \mathrm{ch}\,\theta_1 \end{Vmatrix} \tag{46}$$

Putting

$$k_1 L + \coth\theta_1 = \Lambda_1 \tag{47}$$

$$k_1 p + \coth\theta_1 = \Pi_1 \tag{48}$$

we find, performing the calculations as indicated in Eq. (18)

$$Y_o' = \frac{R}{\mathrm{sh}\,\theta_1 \sqrt{\left(\Lambda_1 \Pi_1 - \frac{1}{\mathrm{sh}^2\theta}\right)^2 + \left(\frac{k_1}{k_2}\,\Pi_1\right)^2}} \tag{49}$$

On the other hand, Eq. (31) gives for the radial acceptance

$$Y_o = \frac{R}{\mathrm{sh}\,\theta_1 \sqrt{\Lambda_1^{\,2} + \left(\frac{k_1}{k_2}\right)^2}} \tag{50}$$

The phase acceptance in the dc plane is therefore

$$A_y = \frac{\pi\, k_1 R^2 / \mathrm{sh}^2\theta_1}{\sqrt{\left[\Lambda_1^{\,2} + \left(\frac{k_1}{k_2}\right)^2\right] \left[\left(\Lambda_1 \Pi_1 - \frac{1}{\mathrm{sh}^2\theta}\right)^2 + \left(\frac{k_1}{k_2}\,\Pi_1\right)^2\right]}} \tag{51}$$

From these relations the angular and radial acceptances, the phase acceptance as well as the total angular acceptance

$$\Omega = \frac{4\,(k_1 R)^2 / \mathrm{sh}\theta_1}{\sqrt{\left(1 + P_1^{\,2}\right)\left[\left(\Lambda_1 \quad _1 - \frac{1}{\mathrm{sh}^2\theta_1}\right)^2 + \left(\frac{k_1}{k_2}\,\Pi_1\right)^2\right]}} \tag{52}$$

can be plotted against whatever parameter is considered to be variable.

Simpler formulae can be obtained in two cases of practical importance:

i. Low Energy Doublet

If one works with low energies, k is high and one has frequently a situation where

$$k_1 p \gg \cot h\theta_1 \tag{53}$$

$$k_1 L \gg \cot h\theta_1 \tag{54}$$

Assuming moreover that k_1 is of the same order of magnitude as k_2, i.e.,

$$k_1 \approx k_2 = k \tag{55}$$

(this would be strictly true for an equigradient doublet) one finds for the partial acceptances

$$X_o' = \frac{R}{p} \tag{56}$$

$$X_o = R \tag{57}$$

$$Y_o' = \frac{R}{p} \, \frac{1}{kL \text{ sh } \theta_1} \tag{58}$$

$$Y_o = R \frac{1}{kL \text{ sh } \theta_1} \tag{59}$$

Eq. (54) shows that the acceptances are much smaller in the dc plane than in the cd plane.

ii. High Energy Doublet

In high energy work k is small and one might have a situation where

$$k_1 p \ll 1 \tag{60}$$

$$k_1 L \ll 1 \tag{61}$$

$$k_1 s \ll 1 \tag{62}$$

Assuming again that k_1 is of the same order of magnitude as k_2, we find for the partial acceptances

$$X_o' = kR \tag{63}$$

$$X_o = R \tag{64}$$

$$Y_o' = kR \frac{1}{\operatorname{ch} \theta_1} \tag{65}$$

$$Y_o = R \frac{1}{\operatorname{ch} \theta_1} \tag{66}$$

If the conditions (60)-(62) are satisfied, the acceptances of the doublet are substantially independent of the source position and the drift length between the quadrupoles. Moreover ch θ_1 being close to unity, the acceptances are essentially the same in the two basic planes, contrary to what happens in low energy work. The explanation of this behavior lies in the fact that a very high energy doublet may work as an extremely weak lens.

In intermediate cases when the conditions (53), (54) or (60)-(62) are not satisfied, one would obviously have to revert to the more accurate formulae given above.

VIII. Application to the Symmetric Triplet

We call a triplet symmetric if

$$k_1 = k_3 = k_o \qquad k_2 = k_i \tag{67}$$

$$s_1 = s_3 = s_o \qquad s_2 = s_i \tag{68}$$

and consequently

$$\theta_1 = \theta_3 = \theta_o \qquad \theta_2 = \theta_i, \tag{69}$$

o standing for outer and i for inner. We assume moreover

$$L_{12} = L_{23} = L \tag{70}$$

In the dcd plane the acceptance can be written without any calculation because in this plane the acceptance of the triplet is that of the doublet made of the first two quadrupoles. Putting, in analogy with the preceding notation,

$$k_o L + \coth \theta_o = \Lambda_o \tag{71}$$

$$k_o p + \coth \theta_o = \Pi_o \tag{72}$$

we can write from Eqs. (49) and (50) for the partial acceptances

$$Y_o' = \frac{k_o R}{\sqrt{\operatorname{sh} \theta_o \left(\Lambda_o \Pi_o - \frac{1}{\operatorname{sh}^2 \theta_o}\right)^2 + \left(\frac{k_o}{k_i} \Pi_o\right)^2}} \tag{73}$$

and

$$Y_o = \frac{R}{\operatorname{sh} \theta_o \sqrt{\Lambda_o^2 + \left(\frac{k_o}{k_i}\right)^2}} \tag{74}$$

and the remarks made in connection with the doublet hold good.

The real character of the triplet is revealed in its cdc plane where the beam envelope can graze the quadrupole aperture either in the first lens or in the third lens, or in both. For an envelope grazing the aperture in the first lens one has

$$x_{o1}' = \frac{k_o R}{\sqrt{1 + (k_o p)^2}} \tag{75}$$

and

$$x_{o1} = R \tag{76}$$

[obviously, the grazing of the aperture does not occur at the same location in the two uses represented by Eqs. (75) and (76).] . For a contact between beam envelope and limiting aperture located in the third lens one has, according to Eqs. (18) and (31) respectively

$$x'_{o3} = \frac{R}{\sqrt{(A_{cdc}p + B_{cdc})^2 + \frac{1}{k_o^2}(C_{cdc}p + A_{cdc})^2}} \tag{77}$$

and

$$x_{o3} = \frac{R}{\sqrt{A_{cdc}^2 + \frac{1}{k_o^2}C_{cdc}^2}} \tag{78}$$

where A_{cdc}, B_{cdc} and C_{cdc} are the elements of the transfer matrix of the complete triplet in its cdc plane.

By means of some slight algebraic manipulations Eqs. (77) and (78) can be written in the form

$$x'_{o3} = \frac{k_o R}{\sqrt{1 + (k_o p)^2 + (k_o^2 B_{cdc} + C_{cdc})(C_{cdc}p^2 + 2A_{cdc}p + B_{cdc})}} \tag{79}$$

$$x_{o3} = \frac{R}{\sqrt{1 + \frac{C_{cdc}}{k_o}(k_o^2 B_{cdc} + C_{cdc})}} \tag{80}$$

which makes them more suitable for comparison with Eqs. (75) and (76) respectively.

It can easily be shown that in the case of a symmetric triplet one always has

$$k_o^2 B_{cdc} + C_{cdc} > 0 \tag{81}$$

The limiting equation for the radial acceptance is therefore

$$C_{cdc} = 0 \tag{82}$$

which simply represents the afocality condition in the cdc plane. For $C_{cdc} > 0$ one has

$$X_o = x_{o3} \tag{83}$$

whereas for $C_{cdc} < 0$ the radial acceptance is

$$X_o = x_{o1} \tag{84}$$

[Eq. (78) shows that the quantity under the root in Eq. (80) can never become negative.]

For $C_{cdc} = 0$ one has

$$X_o = x_{o1} = x_{o3} \tag{85}$$

and the beam envelope grazes the quadrupole aperture at the entrance of the first lens and at the exit of the third lens.

Comparing next Eqs. (75) and (79) one notices that for $C_{cdc}p^2 + 2A_{cdc} + B_{cdc} > 0$ one has

$$X_o' = x_{o3}' \tag{86}$$

whereas for $C_{cdc}p^2 + 2A_{cdc}p + B_{cdc} < 0$ the angular acceptance is

$$X_o' = x_{o1}' \tag{87}$$

[Eq. (77) shows that the quantity under the root in Eq. (79) can never become negative.]

The limiting equation for the angular acceptance is therefore

$$C_{cdc}p^2 + 2A_{cdc}p + B_{cdc} = 0 \tag{88}$$

which is also, as one would expect, the equation for having the object distance equal to the image distance ($p = q$), i.e., complete symmetry. If Eq. (88) is satisfied one has

$$X_o' = x_{o1}' = x_{o3}' \tag{89}$$

and the beam envelope grazes the aperture both in the first and in the third lens.

The solutions of Eq. (88) are

$$p_{(+1)} = H_{cdc} = \frac{1 - A_{cdc}}{C_{cdc}} \tag{90}$$

corresponding to plus unit magnification or to the position of the principal plane, and

$$p_{(-1)} = \overline{H}_{cdc} = -\frac{1 + A_{cdc}}{C_{cdc}} \tag{91}$$

corresponding to minus unit magnification or to the position of the antiprincipal plane.

In the case of unit magnification in the cdc plane of the triplet, the beam excursion has therefore a double maximum in this plane, the first occurring in the first lens and the second in the last lens.

In the case of degeneracy, $C_{cdc} = 0$, the angular acceptance is governed by the first lens or the third lens according to whether $2A_{cdc}p + B_{cdc}$ is negative or positive. The limiting equation reduces here to

$$2A_{cdc}p + B_{cdc} = 0 \qquad (92)$$

and can be satisfied only if $A_{cdc} = -1$.

A more specific discussion of the acceptances in the cdc plane in terms of the focusing parameters of the triplet rather than in terms of the general and formal matrix elements will be given in a separate paper devoted to the theory of the symmetric triplet.

IX. Conclusions

The formulae given above allow us to calculate the angular and phase acceptance of any channel composed of any number of arbitrary quadrupoles if the lens parameters and the drift lengths are given. Conversely, for given (horizontal and vertical) emittances of the beam they allow us to calculate the aperture of the beam channel which will accept the beam emittances. Optimization with respect to the lens parameters is therefore possible by means of a straight forward numerical program.

FOOTNOTES AND REFERENCES

*This work was done under auspices of the U.S. Atomic Energy Commission

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[1] Goldwasser, E. L., and Read, A.L., "Facilities and Plans for the Experimental Program at the 200 GeV Accelerator", National Accelerator Laboratory, September 20, 1967.