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LONGITUDINAL AND TRANSVERSE COUPLING IMPEDANCES OF THE EXTRACTION LAMBERTSONS

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I. Introduction

In a Lambertson magnet, the particle beam is very near to the magnet surface (Fig. 1). As a result, the longitudinal impedance Z_L and the transverse impedances Z_x, Z_y can become very large. The objection of this note is to make an estimate of Z_L and $Z_{x,y}$.

We approximate the cross section of the Lamberson magnet by an annular stack of iron laminations of total length ℓ, inner radius b and outer radius b+d (Fig. 2). The beam is displaced by a distance x_0 from the central axis. Throughout this discussion, the harmonic mode number n of the beam disturbance is restricted to below the pipe cut-off; i.e.,

$$n << \frac{\bar{R}}{b} \sim 0.3 \times 10^5 \qquad (1.1)$$

where $\bar{R} \sim 10^5$ cm is the main radius of the accelerator ring and $b \sim 3$ cm.

The contributions to Z_L and $Z_{x,y}$ due to the resistive magnet core are derived in Sections II, III, IV by assigning to the laminated wall an effective resistivity ρ_{eff}, which is calculated in Section VII. The contributions to the impedances due to space-charge are derived in Sections V and VI. The magnetic and electric field will penetrate into the gaps of the magnetic laminations leading to extra space-charge impedances. However, this effect will be included in the effective resistivity ρ_{eff} and therefore is not discussed in Sections V and VI. Some numerical values for Z_L and $Z_{x,y}$ for different disturbance frequencies are displaced in Section VIII.

II. Longitudinal Impedances Due to the Resistive Walls

Consider a disturbance in the beam which is of harmonic mode n and current I in the positive z direction. Under assumption (1.1), the induced surface current density on the beam wall of the magnet is

$$J(\theta;x_o) = -\frac{I}{2\pi b}\frac{1-g^2}{1+g^2-2g\cos\theta} \tag{2.1}$$

with $g=x_o/b$. Due to the resistivity of the wall, there is an electric field at the wall parallel to $J(\theta;x_o)$,

$$E_z(\theta;x_o) = \mathcal{R}_{eff}\, J(\theta;x_o)\ , \tag{2.2}$$

where the effective surface impedance of the wall is

$$\mathcal{R}_{eff} = \frac{\rho_{eff}}{\delta}\,(1+j) \tag{2.3}$$

and the skin depth

$$\delta = \sqrt{\frac{2\ \rho_{eff}}{\mu\omega}} \tag{2.4}$$

with $\omega=n\beta_W c/\bar{R}$ the angular velocity of the disturbance and $\beta_W c$ its phase velocity. Elsewhere, $E_z(r,\theta;x_o)$ obeys

$$\left[\nabla_\perp{}^2 - \frac{n^2}{\bar{R}^2\gamma^2}\right] E_z(r,\theta;x_o) = 0\ . \tag{2.5}$$

When $n \ll \gamma_W\bar{R}/b$ with $\gamma_W=(1-\beta_W{}^2)^{-1/2}$, we get

$$E_z(r,\theta;x_o) = -\frac{\mathcal{R}_{eff}I}{2\pi b}\ \frac{1-(gr')^2}{1+(gr')^2-2gr'\cos\theta}\ , \tag{2.6}$$

with $r'=r/b$. At the beam position,

$$E_z = -\frac{\mathcal{R}_{eff}I}{2\pi b}\,\frac{1+g^2}{1-g^2}\ ; \tag{2.7}$$

therefore, the longitudinal impedance due to the resistive wall is

$$Z_L = \frac{\mathcal{R}_{eff}\ell}{2\pi b}\,\frac{1+g^2}{1-g^2}\quad . \tag{2.8}$$

This expression can be checked by computing the average power consumed at the wall

$$P = \frac{1}{2}\int_0^{2\pi} \frac{\rho_{eff}\ell}{(b\delta d\theta)}\,|Jbd\theta|^2 \tag{2.9}$$

which should equal

$$\frac{1}{2}(\mathcal{R}e\ Z_L)|I|^2\ .$$

III. Horizontal Transverse Impedance Due to the Resistive Walls

A beam of current I making horizontal oscillations of amplitude Δ and of harmonic frequency n can be represented by a beam dipole, i.e., two line beams of current $\pm I$ at $x_0 \pm \Delta/2$ (Fig. 3). The longitudinal electric field due to the resistivity of the wall is obtained by differentiating Eq. (2.6):

$$\begin{aligned} E_{z\rightarrow}(r,\theta) &= \frac{\partial E_z(r,\theta;x_0)}{\partial x_0}\,\Delta \\ &= \frac{\mathcal{R}_{eff} I\Delta}{\pi b^2}\cdot r' \cdot \frac{2gr' - (1+(gr')^2)\cos\theta}{[1+(gr')^2-2gr'\cos\theta]^2}\ , \end{aligned} \tag{3.1}$$

which produces through Ampere's Law a corresponding transverse magnetic field

$$\vec{B}_{\rightarrow}(r,\theta) = -\frac{1}{j\omega}\vec{\nabla}_{\perp} \times \vec{E}_{z\rightarrow}\ . \tag{3.2}$$

At the beam dipole,

$$B_{r\rightarrow} = 0\ ,$$

$$B_{\theta\rightarrow} = -\frac{\mathcal{R}_{eff}\,\ell}{j\omega\pi b^3}\,\frac{1+g^2}{(1-g^2)^3}\ , \tag{3.3}$$

which displaces the beam in the horizontal direction. The horizontal transverse impedance is defined as

$$Z_x = \frac{j}{\beta_p I\Delta}\int_0^{\ell}\left[E_{x\rightarrow} + (\vec{\beta}_p c \times \vec{B}_{\rightarrow})_x\right]d\ell\ , \tag{3.4}$$

where $\beta_p c$ is the longitudinal velocity of the beam particles. Note that, due to resistivity only, $E_{x\rightarrow} = 0$, thus giving

$$Z_x = \frac{c}{\omega}\,\frac{\mathcal{R}_{eff}\ell}{\pi b^3}\,\frac{1+g^2}{(1-g^2)^3}\ . \tag{3.5}$$

This same Z_x can be obtained also by equating the average power consumed by the beam dipole

$$P = \frac{1}{2}\frac{\omega}{c}\,|I\Delta|^2\,\mathcal{R}e\ Z_x \tag{3.6}$$

to the average power consumed at the walls

$$P = \frac{1}{2}\int_0^{2\pi} \frac{\rho_{eff}}{(\delta b d\theta)} \, |J_{\rightarrow} b d\theta|^2 \ , \tag{3.7}$$

where $J_{\rightarrow}(\theta)$, the surface current density due to the horizontal beam dipole, can be obtained by differentiating Eq. (2.1) with respect to x_o, i.e.,

$$J_{\rightarrow}(\theta) = \frac{\partial J(\theta;x_o)}{\partial x_o} \Delta \ . \tag{3.8}$$

Note that the power consumed by the wall current J_θ has not been included in (3.7) since J_θ is smaller than $J_{\rightarrow}$ in the z-direction by a factor $\frac{nb}{\bar{R}}$.

IV. Vertical Transverse Impedance Due to the Resistive Walls

We now consider the beam of current I making vertical oscillations of amplitude Δ and harmonic frequency n at x_o. This can be represented by a beam dipole with two line beams of currents $\pm I$ at $r=x_o$ and $\theta=\mp\Delta/x_o$ (Fig. 4). The feed back electric field $E_z^{\uparrow}(r,\theta)$ due to the resistive walls is obtained by differentiating Eq. (2.6),

$$E_{z\uparrow}(r,\theta) = \frac{\partial E_z(r,\theta;x_o)}{\partial\theta} \ \frac{\Delta}{x_o}$$

$$= \frac{\mathcal{R}_{eff} I\Delta}{\pi b^2} \ \frac{[1-(gr')^2]r'\sin\theta}{[1+(gr')^2 - 2\,gr'\cos\theta]^2} \tag{4.1}$$

which produces a corresponding transversed magnetic field

$$\vec{B}_{\uparrow} = \frac{1}{j\omega} \ \vec{\nabla}_t \times \vec{E}_{z\uparrow} . \tag{4.2}$$

At the beam position,

$$B_{\theta\uparrow} = 0 ,$$

$$B_{r\uparrow} = - \frac{\mathcal{R}_{eff} I\Delta}{j\omega\pi b^3} \ \frac{1 + g^2}{(1-g^2)^3} \ , \tag{4.3}$$

which displaces the beam in the vertical direction.

The vertical transverse impedance is defined as

$$Z_y = \frac{j}{\beta_p I\Delta}\int_0^{\ell} (\vec{\beta}_p c \times \vec{B}_{\uparrow})_{-y}\, d\ell \ , \tag{4.4}$$

thus giving

$$Z_y = \frac{c}{\omega}\,\frac{\mathcal{R}_{eff}\,\ell}{\pi b^3}\;\frac{1+g^2}{(1-g^2)^3} \ . \tag{4.5}$$

This same Z_y can be obtained by equating the average power consumed by the beam dipole

$$P = \frac{1}{2}\frac{\omega}{c}(I\Delta)^2\,\mathcal{R}e\, Z_y \tag{4.6}$$

to the average power consumed at the wall

$$P = \frac{1}{2}\int_0^{2\pi} \frac{\rho_{eff}}{(\delta b d\theta)}\,(J_{\uparrow}(\theta) b d\theta)^2 \ , \tag{4.7}$$

where $J_{\uparrow}(\theta)$, the surface current density due to the vertical beam dipole, can be obtained by differentiating Eq. (2.1) with respect to θ; i.e.,

$$J_{\uparrow}(\theta) = \frac{\partial J(\theta, x_0)}{\partial\theta}\;\frac{\Delta}{x_0} \ . \tag{4.8}$$

We note that

$$Z_x = Z_y . \tag{4.9}$$

V. Longitudinal Impedance Due to Space Charge

The contribution to Z_L due to space-charge effect can be obtained by computing the longitudinal electric self field at the beam using Ampere's Law:

$$\oint \vec{E}\cdot d\ell = -\oint \dot{\vec{B}}\, da \ , \tag{5.1}$$

The loop is taken to be in the θ=0 plane as shown in Fig. 5. The magnetic field in the loop is

$$B_\theta = \begin{cases} \dfrac{\mu_0 I}{2\pi}\left(\dfrac{x - x_0}{a^2} + \dfrac{1}{x + b^2/x_0^2}\right) & x_0 \leq x \leq x_0 + a \ , \\ \dfrac{\mu_0 I}{2\pi}\left(\dfrac{1}{x - x_0} + \dfrac{1}{x + b^2/x_0^2}\right) & x_0 + a \leq x \leq b \ , \end{cases} \tag{5.2}$$

where a is the radius of the beam. The transverse electric field along the loop is

$$E_x = \begin{cases} \dfrac{\lambda}{2\pi\varepsilon_0}\left(\dfrac{x - x_0}{a^2} + \dfrac{1}{x + b^2/x_0^2}\right) & x_0 \leq x \leq x_0 + a \ , \\ \dfrac{\lambda}{2\pi\varepsilon_0}\left(\dfrac{1}{x - x_0} + \dfrac{1}{x + b^2/x_0^2}\right) & x_0 + a \leq x \leq b \ , \end{cases} \tag{5.3}$$

where the disturbed beam line charge density is related to the disturbed current I by

$$I = \lambda \beta_w c \ . \tag{5.4}$$

Putting everything into (5.1), we get for the self field at the beam

$$E_z = \frac{j\omega Z_0 I}{2\pi c \beta_w^2 \gamma_w^2}\left(\frac{1}{2} + \ell n\left(\frac{b}{a} \cdot \frac{1 - g^2}{1 + g^2}\right)\right) \tag{5.5}$$

leading to

$$Z_L/n = -j\, Z_0 \, \frac{\ell}{2\pi R} \, \frac{1}{\beta_w \gamma_w^2}\left(\frac{1}{2} + \ell n \, \frac{b}{a} \cdot \frac{1 - g^2}{1 + g^2}\right) \tag{5.6}$$

where $Z_0 = 120\pi$ ohms is the impedance of free space.

VI. Transverse Impedance Due to Space-Charge

The contribution to the transverse impedance due to space charge consists of two parts: the image current effect and the beam size effect. Let us first study the magnetic contribution. The image current due to a horizontal beam dipole at x_0 is (Eq. (3.8))

$$J_{\rightarrow}(\theta) = \frac{I\Delta}{\pi b^2} \frac{2g - (1+g^2)\cos\theta}{(1+g^2-2g\cos\theta)^2} \tag{6.1}$$

which produces at the beam position a vertical magnetic field

$$B_y = \frac{\mu_o I\Delta}{2\pi b^2} \frac{1}{1 - g^2} \quad . \tag{6.2}$$

Note that this B_y is different from the one discussed in Section III where the magnetic field arises solely from the resistivity of the walls.

The actual unperturbed beam I has a finite radius a >> Δ centered at x_o and carrying a current density

$$J(R,\phi;x_o) = \frac{I}{\pi a^2} \theta(a-R) \tag{6.3}$$

with $R = \sqrt{r^2+x_o^2-2rx_o\cos\theta}$, the radial distance from the beam center as shown in Fig. 6. A horizontal disturbance of amplitude Δ gives a perturbing current density

$$J'(R,\phi;x_o) = \Delta \frac{\partial J(R,\phi;x_o)}{\partial x_o}$$

$$= \frac{I\Delta}{\pi a^2} \delta(R-a) \cos\theta \tag{6.4}$$

which resides solely on the beam surface and produces at the beam a vertical magnetic self field

$$B_y = - \frac{\mu_o I\Delta}{2\pi a^2} \quad . \tag{6.5}$$

Combining Eq. (6.2), (6.5) and using Eq. (3.4), we obtain for the magnetic contribution

$$Z_n = \frac{JZ_o \ell}{2\pi} \left[\frac{1}{a^2} - \frac{1}{b^2(1-g^2)^2}\right] . \tag{6.6}$$

The contribution due to electric field can easily be included to give

$$Z_x = -\frac{jZ_o\ell}{2\pi}\left[\frac{1}{\beta_p^2} - 1\right]\left[\frac{1}{a^2} - \frac{1}{b^2(1-g^2)^2}\right]$$

$$= -\frac{jZ_o\ell}{2\pi\beta_p^2\gamma_p^2}\left[\frac{1}{a^2} - \frac{1}{b^2(1-g^2)^2}\right] . \tag{6.7}$$

Exactly the same result is obtained for Z_y.

Summing up, we have the total coupling longitudinal and transverse impedances of the Lambertson magnets:

$$Z_L = \frac{\ell \mathcal{R}_{eff}}{2\pi b}\ \frac{1+g^2}{1-g^2} - jZ_o\ \frac{n\ell}{2\pi\overline{R}}\ \frac{1}{\beta_w\gamma_w^2}\ \left(\frac{1}{2} + \ln\left(\frac{b}{a}\cdot\frac{1-g^2}{1+g^2}\right)\right) , \tag{6.8}$$

$$Z_{x,y} = \frac{\overline{R}\ell}{\pi n\beta_w b^3}\mathcal{R}_{eff}\ \frac{1+g^2}{(1-g^2)^3} - jZ_o\ \frac{\ell}{2\pi\beta_p^2\gamma_p^2}\left(\frac{1}{a^2} - \frac{1}{b^2(1-g^2)^2}\right) . \tag{6.9}$$

VII. Effective Resistivity of the Laminated Walls

When the perturbed frequency is low enough, the skin depth δ may be bigger than half the lamination thickness τ. This occurs when

$$\frac{\omega}{2\pi} < \frac{2\rho}{\mu_R Z_o}\ \frac{4}{\tau^2} = \frac{750}{\mu_R}\ \text{kHz}, \tag{7.1}$$

where we have taken τ~1 mm, the resistivity of iron ρ=73$\mu\Omega$. If the relative permeability of iron μ_R taken as ~300 at low frequencies, this situation can occur only for the "slow wave" of the transverse disturbance*. When this happens the skin depth δ is set to equal the thickness of the magnetic iron core d to obtain an effective resistivity ρ_{eff}

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*In considering the stability due to a disturbance, through dispersion relations, $\omega \sim n\omega_0$ for the longitudinal case while $\omega \sim (n-\nu)\omega_0$ with $n>\nu$ for the transverse case, where ω_0 is the longitudinal angular velocity of the beam particles and ν the natural betatron tones. For a $\overline{R}$~1 km machine, $\omega_0/2\pi$ ~ 48 kHz.

$$\rho_{eff} = \frac{d^2 \mu_R Z_o \omega}{2c}$$

which is to be substituted in Eq. (6.8) and (6.9) for the evaluation of Z_L and $Z_{x,y}$.

When $\omega/2\pi > \frac{750}{\mu_R}$ kHz, the wall current is always confined to the surface of the wall (i.e., the edges of the laminations) and the surfaces of the gaps (i.e., the surfaces of the laminations).

The resistivity of iron leads to an impedance for the edges of the laminations

$$Z_{edge} = \frac{\mathcal{R}\ell}{2\pi b}, \tag{7.2}$$

where R = $(\rho/\delta)(1+j)$, and an impedance for the surfaces of the gaps

$$Z_{gap} = \frac{\mathcal{R}}{\pi} \ell n(1+\frac{d}{b}) \frac{\ell}{\tau}, \tag{7.3}$$

where ℓ/τ is the number of gaps, giving an effective surface impedance (aside from the geometry of the gap)

$$\mathcal{R}_{eff} = \mathcal{R} \left[1 + \frac{2b}{\tau} \ell n \ (1+\frac{d}{b}) \right]. \tag{7.4}$$

Taking 2b ~ d ~ 6 cm, we see that, due to path length difference, Z_{edge} is smaller than Z_{gap} by 66 times and can therefore be neglected safely. In addition, each gap has a capacitance per unit "length" $\frac{\varepsilon_0 2\pi r}{\Delta_0}$, Δ_0 ~ .03 mm being the gap width, giving a capacitance of

$$C = \frac{\varepsilon_o \pi d^2}{\Delta_o} \left(1 + \frac{2b}{d} \right) \tag{7.5}$$

for each gap. There is a corresponding inductance per unit "length" $\frac{\mu_0 \Delta_0}{2\pi r}$, giving an inductance of

$$L = \frac{\mu_0 \Delta_0}{2\pi} \ell n \left(1 + \frac{d}{b} \right) \tag{7.6}$$

for each gap. The gap, considered as a short-circuited transmission line, therefore has a reactance

$$Z = \left(\frac{\mathcal{R}}{\pi} + \frac{j\omega\mu_o\Delta_o}{2\pi} \right) \ln \left(1 + \frac{d}{b} \right) \tag{7.7}$$

and an admittance

$$Y = \frac{j\omega\varepsilon_o\pi d^2}{\Delta_o} \left(1 + \frac{2b}{d} \right) \cdot \tag{7.8}$$

Since b/d ~ O(1), the characteristic impedance of the line can be approximated by

$$Z_c = \sqrt{Z/Y} \ , \tag{7.9}$$

giving an input impedance of

$$Z_{in} = Z_c \tanh\sqrt{ZY} \ . \tag{7.10}$$

Following Eq. (7.4), the total effective surface impedance of the laminated walls is

$$\mathcal{R}_{eff} = \frac{2\pi b}{\tau}\sqrt{\frac{Z}{Y}} \tanh\sqrt{ZY} \ , \tag{7.11}$$

which is the value for $\mathcal{R}_{eff}$ to be substituted into the formulae (6.8) and (6.9) for Z_L and $Z_{x,y}$.

When $\sqrt{ZY} << 1$, $Z_{in}=Z$, and we have

$$\mathcal{R}_{eff} = \frac{2\pi b}{\tau} \left(\frac{\mathcal{R}}{\pi} + \frac{j\omega\mu_o\Delta_o}{2\pi} \right) \ln \left(1 + \frac{d}{b} \right) \cdot \tag{7.12}$$

This is exactly equivalent to the condition that the length d of the short-circuited line is much less than the reduced wavelength c/ω of the disturbance; i.e.,

$$\frac{\omega}{2\pi} << \frac{c}{2\pi d} \approx 0.80 \text{ GHz} \tag{7.13}$$

corresponding* to $n << \bar{R}/d \approx 17{,}000$. Numerically, Eq. (7.12) gives

$$\mathcal{R}_{eff} = [.024(1+j)\sqrt{n\mu_R} + .00037\, jn] \text{ ohms.} \tag{7.14}$$

*For transverse disturbances, in the followings, all n should be replaced by $n-\nu$ unless when $n >> \nu$.

Again, the first term represents the resistivity of iron and the second the geometry of the gap. The two terms are equal in magnitude when $\delta = \Delta_o/2\sqrt{2\mu_R}$, or when $\frac{\omega}{2\pi} \approx 0.43\ \mu_R$ GHz ($n=8900\ \mu_R$). For such a high frequency, μ_R can be taken as close to unity. Thus when $n \leq 100 << 8900$, only the resistive part contributes and

$$\mathcal{R}_{eff} = .024\ (1+j)\sqrt{n\mu_R}\ \text{ohms}\ , \qquad (7.15)$$

whereas when $100 \leq n << 17{,}000$, Eq. (7.14) has to be used.

For still higher frequencies, we meet with resonances. The pth resonance occurs when

$$\mathcal{I}m\sqrt{YZ} = (2p-1)\frac{\pi}{2}\ . \qquad (7.16)$$

At resonance,

$$Z_{in} = Z_c \coth \mathcal{R}e\sqrt{YZ} \qquad (7.17)$$

and the figure of merit is ($\Delta\omega$ is FWHM)

$$Q \equiv \frac{\omega_{Res}}{\Delta\omega} = \pi \coth \mathcal{R}e\sqrt{YZ}\ . \qquad (7.18)$$

Since the resistive term in Eq. (7.14) becomes negligible only when $n \gtrsim 890{,}000 >> 8900$, we expect θ to be small or the resonances not sharp at all. Putting in numerical values,

$$\sqrt{YZ} = \frac{j\omega d}{c}\sqrt{\frac{1}{2}\left(1+\frac{2b}{d}\right)\ln\left(1+\frac{d}{d}\right)}\sqrt{1+\frac{\mu_R\delta}{\Delta_o}(1-j)}$$

$$= j\ (6.3 \times 10^{-5}n)\sqrt{1+67\sqrt{\frac{\mu_R}{n}}(1-j)} \qquad (7.19)$$

with $\mu_R \sim 1$, we get for the first three resonances

n_{res}	Q	$\mathcal{R}_{eff}$ at resonance
20,000	3.3	(30-4.7j) ohms
66,000	5.3	(15-1.5j) ohms
113,000	6.9	(9.7-.85j) ohms

If ℓ, the length of the Lambertson magnets, is ~20 m, the laminated walls give the following contribution

$n \lesssim 100 << 8,900$,

$$Z_L/G_L(x_o) = 2.5\ (1+j)\ \sqrt{\frac{n}{\mu}}\ \Omega\ ,$$

$$Z_{x,y}/G_{x,y}(x_o) = 5.6 \times 10^4\ (1+j)\sqrt{\frac{\mu_R}{n}}\ \Omega/\text{cm}\ ,$$

$n \lesssim 2,000 << 17,000$,

$$Z_L/G_L(x_o) = [2.5\ (1+j)\ \sqrt{n\mu_R} + .039j\ n]\ \Omega\ ,$$

$$Z_{x,y}/G_{x,y}(x_o) = [5.6 \times 10^4\ (1+j)\sqrt{\frac{\mu_R}{n}} + 860j]\ \Omega/\text{cm}.$$

Resonances:

n_{res}	Q	$[Z_L/G_L(x_o)]_{res}$	$[Z_{x,y}/G_{x,y}(x_o)]_{res}$
20,000	3.3	(3200-500j) Ω	(3500-550j) Ω/cm.
66,000	5.3	(1600-160j) Ω	(530-53j) Ω/cm.
113,000	6.9	(1020-90j) Ω	(200-18j) Ω/cm.

For $n \gtrsim 2,000$, formula (7.10) has to be used. In above

$$G_L(x_o) = \frac{1 + x_o^2/b^2}{1 - x_o^2/b^2}\ , \qquad G_{x,y}(x_o) = \frac{1 + x_o^2/b^2}{(1 - x_o^2/b^2)^3}$$

are the beam displacement factors.