national accelerator laboratory

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BOOSTER CONSTANT BUCKET AREA REGIME WITH SPACE CHARGE

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PURPOSE

To study further the synchrotron motion discussed by Gumowski and Reich.[1] In particular, a hamiltonian approach is used and applied to the determination of a radio frequency regime to maintain constant bucket size.

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GENERAL

Symon and Sessler[2] give the basis for a hamiltonian formulation of the longitudinal motion assuming that coupling between betatron and synchrotron motion may be neglected. A particle with energy E travels along an orbit of length $2\pi R(E)$ where $R(E)$ is the equivalent radius. An equivalent angular variable θ is defined such that

$$ds = R d\theta, \tag{1}$$

where ds is an element of arc length along the orbit. If $\mathcal{E}$ is the electric field

$$\frac{dE}{ds} = e\mathcal{E}. \tag{2}$$

In addition an angular frequency ω is introduced so that

$$\frac{d\theta}{dt} = \omega. \tag{3}$$

Following Symon and Sessler[2] a canonical variable is introduced (here a more customary definition is used involving angular frequency instead of revolution frequency)

$$P \equiv \int \frac{dE}{\omega} . \tag{4}$$

The equations of motion then become

$$\dot{P} = eR\mathcal{E} \tag{5}$$

$$\dot{\theta} = \omega. \tag{6}$$

SINGLE GAP EXCITATION AND SPACE CHARGE EFFECTS

Symon and Sessler[2] treat the case of multiple gaps. Here, for simplicity only a single gap is introduced. Thus

$$E = \frac{V}{R}\,\delta(\theta)\,\sin\int^{t}\omega_{RF}dt - \frac{c^2 g_0}{R^2\gamma^2}\cdot\frac{d\lambda(\theta)}{d\theta}\,, \quad \text{(emu)} \tag{7}$$

where the approximation for the space charge electric field mentioned in Gumowski and Reich[1] has been used. According to Nielsen and Sessler[3]

$$g_0 = 1 + 2\,\ln\left(\frac{2G}{\pi a}\right) \tag{8}$$

for a beam of radius a between plane parallel plates whose gap is G.

$$\delta(\theta) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{1}^{\infty}\cos n\theta \tag{9}$$

and

$$\cos n\theta\,\sin\int^{t}\omega_{RF}dt = \frac{1}{2}\left\{\sin\left(n\theta+\int^{t}\omega_{RF}dt\right) - \sin\left(n\theta-\int^{t}\omega_{RF}dt\right)\right\}\,. \tag{10}$$

For convenience the radio frequency angular frequency is written as

$$\omega_{RF} \equiv h\omega_s \tag{11}$$

where h is the harmonic number. Since it is expected that $\dot{\theta} = \omega \simeq \omega_s$, the only term that is approximately stationary gives for the electric field

$$E = -\frac{V}{2\pi R}\,\sin\left(h\theta-h\int^{t}\omega_s dt\right) - \frac{c^2 g_0}{R^2\gamma^2}\cdot\frac{d\lambda(\theta)}{d\theta}\,. \tag{12}$$

The equations of motion, Eqs. (5) and (6) become

$$\dot{P} = -\frac{eV}{2\pi} \sin\left(h\theta - h\int^t \omega_s dt\right) - \frac{ec^2 g_0}{R\gamma^2} \cdot \frac{d\lambda(\theta)}{d\theta} \tag{13}$$

$$\dot{\theta} = \omega = \frac{dE}{dP}. \tag{14}$$

Partial integration yields the hamiltonian

$$H = E(P) - \frac{eV}{2\pi h} \cos\left(h\theta - h\int^t \omega_s dt\right) + \frac{ec^2 g_0}{R\gamma^2} \cdot \lambda(\theta). \tag{15}$$

TRANSFORMATION FROM LABORATORY SYSTEM (P,θ) TO (W,ϕ)

A rotating coordinate system is introduced through the generator

$$S(W,\theta) = hW\left(\theta - \int^t \omega_s dt\right). \tag{16}$$

Then

$$P = \frac{\partial S}{\partial \theta} = hW, \tag{17}$$

$$\phi = \frac{\partial S}{\partial W} = h\theta - h\int^t \omega_s dt. \tag{18}$$

The new hamiltonian is given by

$$K = H + \frac{\partial S}{\partial t} = E(P) - h\omega_s W - \frac{eV}{2\pi h} \cos\phi + \frac{ec^2 g_0 h}{R\gamma^2} \lambda(\phi) \tag{20}$$

where $\lambda(\phi)$ follows from $\lambda(\theta)d\theta = \lambda(\phi)d\phi$.

The equations of motion become

$$\dot{W} = -\frac{eV}{2\pi h} \sin\phi - \frac{ec^2 g_0 h}{R\gamma^2} \cdot \frac{d\lambda(\phi)}{d\phi}, \tag{21}$$

$$\dot{\phi} = h\frac{dE}{dP} - h\omega_s. \tag{22}$$

REFERENCE MOTION

Choose

$$\dot{W}_R = -\frac{eV}{2\pi h} \sin \phi_R \tag{23}$$

$$\phi_R = h\left(\omega_R - \omega_s\right) \tag{24}$$

$$\omega_R = \left(\frac{dE}{dP}\right)_{P=hW_R}. \tag{25}$$

SYNCHROTRON MOTION

Motion about the reference motion may be found from the generator

$$S(J,\phi) = \phi(W_R+J) . \tag{26}$$

Thus

$$W = \frac{\partial S}{\partial \phi} = W_R+J, \tag{27}$$

$$\phi = \frac{\partial S}{\partial J} = \text{new coordinate.} \tag{28}$$

If E(P) is expanded

$$E(P) = E_R + \left(\frac{dE}{dP}\right)_R \Delta P + \frac{1}{2}\left(\frac{d^2E}{dP^2}\right)_R \Delta P^2 + \frac{1}{6}\left(\frac{d^3E}{dP^3}\right)_R \Delta P^3 + \ldots . \tag{29}$$

Then, with $\Delta P = hJ$, the new hamiltonian becomes

$$\begin{aligned} \mathcal{R} = E_R &+ \omega_R hJ + \frac{1}{2}\,\omega_R' h^2 J^2 + \frac{1}{6}\,\omega_R'' h^3 J^3 - \omega_s h(W_R+J) \\ &- \frac{eV}{2\pi h}(\cos\phi + \phi \sin\phi_R) + \frac{ec^2 g_0 h}{R\gamma^2} \cdot \lambda(\phi) \end{aligned} \tag{30}$$

where ω_R, ω_R' are clear from Eqs. (25) and (29).

The terms depending only on time may be altered without changing the motion. Hence if

$$\phi_2 = \pi - \phi_R, \tag{31}$$

the hamiltonian may be written as

$$\mathcal{R} = \frac{1}{6}\,\omega_R'' h^3 J^3 + \frac{1}{2}\,\omega_R' h^2 J^2 + (\omega_R - \omega_s) h J$$
$$- \frac{eV}{2\pi h}\left[\cos\phi - \cos\phi_2 + (\phi - \phi_2)\,\sin\phi_R\right] + \frac{ec^2 g_0 h}{R\gamma^2} \cdot \lambda(\phi). \tag{32}$$

ADIABATIC APPROXIMATION

Although the hamiltonian $\mathcal{R}$ contains the time, if these time varying functions are slowly varying with respect to the revolution period of the J-ϕ motion, then the hamiltonian may be treated as a constant of the motion. In particular if ϕ_R, V, ω_R' are constants then Eq. (24) gives $\omega_R = \omega_s$ and

$$\mathcal{R} = \frac{1}{6}\omega_R'' h^3 J^3 + \frac{1}{2}\omega_R' h^2 J^2 - \frac{eV}{2\pi h}\left[\cos\phi - \cos\phi_2 + (\phi - \phi_2)\,\sin\phi_R\right]$$
$$+ \frac{ec^2 g_0 h}{R\gamma^2} \cdot \lambda(\phi). \tag{33}$$

FIXED POINTS

For a conservative hamiltonian fixed points for the motion occur at

$$\dot{J} = -\frac{\partial R}{\partial \phi} = \frac{eV}{2\pi h}(\sin\phi - \sin\phi_2) - \frac{ec^2 g_0 h}{R\gamma^2} \cdot \frac{d\lambda(\phi)}{d\phi} = 0, \tag{34}$$

$$\dot{\phi} = \frac{\partial R}{\partial J} = \frac{1}{2}\,\omega_R'' h^3 J^2 + \omega_R' h^2 J = 0. \tag{35}$$

Reasonable space charge distributions $\lambda(\phi)$ are not expected to yield an electric field at the fixed points. Thus if $\lambda'(\phi)$ is zero at the points determined by Eqs. (34) and (35) the fixed points are:

	J	ϕ	J	ϕ
Stable	0	ϕ_R	$-\frac{2\omega_R'}{h\omega_R''}$	$\pi-\phi_R$
Unstable	0	$\pi-\phi_R$	$-\frac{2\omega_R'}{h\omega_R''}$	ϕ_R

Transition energy occurs at $\omega_R' = 0$. Below transition $\omega_R' > 0$ and $0 \le \phi_R \le \pi/2$. Above transition $\omega_R' < 0$ and $\pi/2 \le \phi_R \le \pi$. Far from transition the bucket characterized by the second pair of J-ϕ points is outside of the vacuum chamber. Near transition, however, both buckets are needed to specify the motion.

Following the usual custom, the term in ω_R'' will be dropped for general simplicity. Adiabatic solutions will be used below transition and above transition. A nonadiabatic change in the phase ϕ_R is made through transition to connect the two solutions. For a linearized treatment passing through transition in which ω_R'' is omitted, see Courant and Snyder.[4]

The effect of the ω_R'' term is considered by Symon and Sessler[2] and by Kolomensky and Lebedev.[5]

SPACE CHARGE DISTRIBUTION

Gumowski and Reich[1] consider several forms for the space charge distribution. In this note only the case of a uniform distribution in the J-ϕ space will be considered.
Thus

$$\lambda(\phi) = 2\sigma J_b(\phi) \tag{36}$$

where J_b is the value of J on the separatrix. Equation (33) becomes

$$R = \frac{1}{2}\omega_R' h^2 J^2 - \frac{eV}{2\pi h}\left[\cos\phi - \cos\phi_2 + (\phi-\phi_2)\sin\phi_R\right] + \frac{2ec^2 g_0 \sigma h}{R\gamma^2} J_b(\phi). \tag{37}$$

BUCKET

On the separatrix $R = 0$. Therefore, if one sets

$$y = \sqrt{\frac{\pi|\omega_R'|h^3}{eV}}\, J, \tag{38}$$

and

$$S = \frac{2\pi c^2 g_0 \sigma h}{R\gamma^2}\sqrt{\frac{e}{\pi h|\omega_R'|V}}, \tag{39}$$

Eq. (37) becomes

$$y^2 + 2Sy - \left[\cos\phi - \cos\phi_2 + (\phi-\phi_2)\sin\phi_R\right] = 0. \tag{40}$$

If one defines the bucket factor

$$\alpha(\phi_R,S) = \frac{1}{4\sqrt{2}} \int_{\phi_1}^{\phi_2} y d\phi, \tag{41}$$

where ϕ_1 is a solution of Eq. (40) for y = 0, then using Eq. (39) one has for the number of particles in the bucket.

$$N = \frac{h}{e} \int_{\phi_1}^{\phi_2} \lambda(\phi) d\phi = \frac{4\sqrt{2}}{\pi h} \cdot \frac{RV}{ec^2} \gamma^2 S\alpha(\phi_R,S). \quad \text{(emu)} \tag{42}$$

CONSTANT AREA BUCKETS

For the reference particle

$$V \sin \phi_R = 2\pi R^2 \langle \dot{B} \rangle . \quad \text{(emu)} \tag{43}$$

Hence the relation for N becomes

$$N = \frac{4\sqrt{2}}{\pi h} \cdot \frac{R}{r_p} \cdot \frac{2\pi e R^2 \langle \dot{B} \rangle}{M_0 c^2} \gamma^2 S\alpha(\phi_R,S), \quad \text{(emu)} \tag{44}$$

where r_p is the classical proton radius.

If the momentum spread of the injected coasting beam is $\pm\Delta p$, then the bucket area required to contain this beam is

$$A = 4\pi R\Delta p = 4\pi \frac{R}{c} \cdot pc \cdot \frac{\Delta p}{p} . \quad \text{(erg-sec)} \tag{45}$$

In a constant bucket area regime this area must equal that of h accelerating buckets. Thus

$$A = \frac{8}{\omega_R} \sqrt{\frac{2eM_0c^2\gamma_R V}{\pi h |\kappa_R|}}\, \alpha(\phi_R,S), \quad \text{(erg-sec)} \tag{46}$$

where

$$\kappa_R = \frac{\gamma_{tran}^2 - \gamma_R^2}{\gamma_{tran}^2\left(\gamma_R^2 - 1\right)} \quad . \tag{47}$$

Thus

$$\frac{A}{M_0 c R} = \frac{16}{\sqrt{2\pi h}}\, \gamma_{tran} \sqrt{\frac{2\pi e R^2 \langle \dot{B} \rangle}{M_0 c^2} \cdot \frac{\gamma_R^3}{\left|\gamma_{tran}^2 - \gamma_R^2\right|}} \cdot \frac{\alpha(\phi_R, S)}{\sqrt{\sin\phi_R}} \quad . \tag{48}$$

(emu)

Given the magnetic cycle $\langle B \rangle$ one may find γ_R from

$$\gamma_R^2 = 1 + \left(\frac{eR\langle B \rangle}{M_0 c}\right)^2 . \qquad \text{(emu)} \tag{49}$$

Thus for a constant A given by Eq. (45) Eqs. (44) and (48) may be solved simultaneously for ϕ_R and S at each instant of time. Eq. (43) then yields the required volts per turn V.

RESULTS

The appropriate equations have been coded for numerical evaluation in the program BUCKET. Results for the booster are shown in the attached computer output. Ignore the columns labeled Power and Res or see TM-302 for explanation. Above transition S is replaced by -S in Eq. (39). Outputs for 3.85×10^{12} particles and a negligible number of particles are shown. Thus the effect of space charge on the required volts per turn (VPT) is rather small, about 2 percent at most.

REFERENCES

1. I. Gumowski and K.H. Reich, Synchrotron Motion in the Presence of Space Charge, CERN/SI/Int. DL/70-6, July 30, 1970.

2. K.R. Symon and A.M. Sessler, Methods of Radio Frequency Acceleration in Fixed Field Accelerators with Applications to High Current and Intersecting Beam Accelerators. MURA-106, April 16, 1956.

3. C.E. Nielsen and A.M. Sessler, Longitudinal Space Charge Effects in Particle Accelerators, Rev. Sci. Instr. 30, 80 (1959).

4. E.D. Courant and H.S. Snyder, Theory of the Alternating Gradient Synchrotron, Annals of Physics 3, 1 (1958).

5. A.A. Kolomensky and A.N. Lebedev, Theory of Cyclic Accelerators, John Wiley and Sons, Inc., New York, 1966, p. 196.

CONSTANT AREA BUCKETS FOR BOOSTER 8 GEV

RADIUS(M)=	75.47170	DELP/P =	.00080	KETRAN(GEV)=	4.17150	AVBINJ(KG)	.28482	AVBMAX(KG)=	3.92859
FREQ(HZ) =	15.00000	HARM. NO.=	84.00000	NO. POINTS=	40.00000	FACTOR	1.00000	GAP(IN) =	2.05000
HBMDIA(IN)=	3.00000	VBMDIA(IN)=	1.25000	PART(E+12)=	3.85000				

NIT	TIME (MSEC)	KE (GEV)	RADFREQ (MHZ)	PHTS (DEGREES)	VPT (MV)	BUCKET ALPHA	S	POWER (KW)	RES
5	.00083	.207242	30.465166	16.5667	.169100	.5335	.0316	.9644	.02260
4	.00167	.229483	31.615038	22.7443	.249653	.4270	.0270	1.3505	.02334
4	.00250	.268143	33.381552	27.0953	.316078	.3606	.0253	1.8361	.02446
4	.00333	.325220	35.566339	30.7787	.371101	.3091	.0246	2.4912	.02576
4	.00417	.402832	37.944195	34.2241	.413134	.2652	.0245	3.3465	.02705
3	.00500	.502778	40.306320	37.5704	.457563	.2263	.0246	4.4022	.02814
3	.00583	.626208	42.494780	40.8531	.490876	.1916	.0247	5.6346	.02894
3	.00667	.773505	44.416922	44.0546	.513475	.1611	.0249	7.0119	.02944
3	.00750	.944313	46.039536	47.1380	.544489	.1346	.0250	8.5076	.02973
3	.00833	1.137682	47.371537	50.0685	.566685	.1120	.0251	10.1084	.02986
3	.00917	1.352223	48.445104	52.8229	.586485	.0929	.0253	11.8157	.02990
3	.01000	1.586254	49.301112	55.3923	.604048	.0770	.0256	13.6470	.02989
3	.01083	1.837904	49.980213	57.7802	.619350	.0637	.0260	15.6421	.02986
3	.01167	2.105189	50.518428	60.0002	.632259	.0525	.0267	17.8776	.02982
3	.01250	2.386050	50.945999	62.0736	.642583	.0431	.0278	20.5032	.02978
3	.01333	2.678389	51.286892	64.0287	.650108	.0352	.0294	23.8386	.02975
3	.01417	2.980080	51.560138	65.9019	.654604	.0282	.0319	28.6690	.02972
3	.01500	3.288984	51.780468	67.7444	.655824	.0220	.0360	37.4068	.02963
3	.01583	3.602958	51.959253	69.6394	.653467	.0162	.0437	60.9252	.02966
4	.01667	3.919865	52.105256	71.7593	.647010	.0099	.0641	217.6944	.02964
4	.01750	4.237578	52.225243	92.0350	.613045	.0048	.1252	3749.3960	.02963
4	.01833	4.553993	52.324432	102.2293	.622061	.0107	.0503	141.0841	.02961
3	.01917	4.867036	52.406899	104.4431	.617055	.0135	.0365	70.1513	.02960
3	.02000	5.174670	52.475814	105.6665	.607006	.0154	.0299	52.1526	.02959
3	.02083	5.474910	52.533669	106.4889	.592105	.0168	.0260	44.5012	.02958
3	.02167	5.765829	52.582426	107.1024	.572886	.0178	.0234	40.3446	.02957
3	.02250	6.045567	52.623546	107.5987	.549703	.0188	.0215	37.7170	.02956
3	.02333	6.312343	52.658570	108.0306	.522844	.0197	.0203	35.8649	.02956
3	.02417	6.564465	52.688193	108.4306	.492560	.0206	.0194	34.4282	.02955
2	.02500	6.800339	52.713313	108.8248	.459098	.0215	.0189	33.2286	.02955
2	.02583	7.018473	52.734575	109.2334	.422701	.0226	.0186	32.1534	.02955
3	.02667	7.217494	52.752498	109.6760	.383612	.0238	.0187	31.1297	.02954
3	.02750	7.396150	52.767502	110.1738	.342079	.0253	.0190	30.0993	.02954
3	.02833	7.553318	52.779922	110.7549	.298354	.0271	.0196	29.0161	.02954
3	.02917	7.688011	52.790030	111.4592	.252688	.0295	.0208	27.8283	.02954
3	.03000	7.799385	52.798037	112.3517	.205328	.0327	.0225	26.4682	.02954
3	.03083	7.886740	52.804108	113.5555	.156501	.0375	.0253	24.8252	.02954
4	.03167	7.949531	52.808367	115.3568	.106383	.0455	.0304	22.6826	.02953
4	.03250	7.987363	52.810908	118.7226	.054981	.0632	.0419	19.4077	.02953

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FREQ(HZ) = 15.00000 HARM. NO.= 84.00000 NO. POINTS= 40.00000 FACTOR = 1.00000 GAP(IN) = 2.05000
HRMDIA(IN)= 3.00000 VRMDIA(IN)= 1.25000 PART(E+12)= .00100

NTT	TIME (MSEC)	KE (GEV)	RADFREQ (MHZ)	PHIS (DEGREES)	VOT (MV)	BUCKET ALPHA	S	POWER (KW)	RES
6	.00083	.207242	30.465167	17.2119	.162943	.5435	.0000	.0000	.02260
4	.00167	.229483	31.615038	23.4247	.241821	.4330	.0000	.0000	.02334
4	.00250	.268143	33.381553	27.7970	.307629	.3649	.0000	.0000	.02445
4	.00333	.325220	35.566339	31.5318	.363119	.3125	.0000	.0000	.02576
4	.00417	.402832	37.944196	35.0266	.403738	.2679	.0000	.0000	.02705
3	.00500	.502778	40.306321	38.4267	.448893	.2285	.0000	.0000	.02814
3	.00583	.626209	42.494730	41.7650	.482066	.1934	.0000	.0000	.02894
3	.00667	.777505	44.416922	45.0219	.510638	.1625	.0000	.0000	.02944
3	.00750	.944313	46.039536	48.1601	.535706	.1357	.0000	.0000	.02973
3	.00833	1.137682	47.371537	51.1456	.553003	.1129	.0000	.0000	.02986
3	.00917	1.352223	48.445104	53.9568	.577925	.0936	.0000	.0000	.02990
3	.01000	1.596254	49.301112	56.5871	.595608	.0775	.0000	.0000	.02983
3	.01083	1.837904	49.980214	59.0436	.611008	.0641	.0000	.0000	.02986
3	.01167	2.105189	50.518498	61.3447	.623977	.0529	.0000	.0000	.02982
3	.01250	2.385050	50.946000	63.5186	.634307	.0434	.0000	.0000	.02978
3	.01333	2.678389	51.286892	65.6043	.641755	.0354	.0000	.0000	.02975
2	.01417	2.980080	51.560138	67.6576	.645054	.0284	.0000	.0000	.02972
3	.01500	3.289984	51.798468	69.7684	.645878	.0222	.0000	.0000	.02963
3	.01583	3.602958	51.959253	72.1187	.643735	.0163	.0000	.0000	.02966
4	.01667	3.919865	52.105256	75.2804	.635386	.0100	.0000	.0000	.02964
5	.01750	4.237578	52.225237	100.9470	.623993	.0048	.0000	.0000	.02963
4	.01833	4.553993	52.324428	105.0841	.628627	.0107	.0000	.0000	.02961
3	.01917	4.867036	52.406895	106.5696	.623443	.0135	.0000	.0000	.02960
3	.02000	5.174570	52.475809	107.4434	.612629	.0153	.0000	.0000	.02959
3	.02083	5.474910	52.533652	108.0475	.597134	.0167	.0000	.0000	.02958
3	.02167	5.765929	52.582419	108.5120	.577431	.0178	.0000	.0000	.02957
3	.02250	6.045567	52.623637	108.9021	.553843	.0187	.0000	.0000	.02956
3	.02333	6.312343	52.658560	109.2568	.526633	.0196	.0000	.0000	.02956
3	.02417	6.564465	52.688181	109.6025	.496044	.0205	.0000	.0000	.02955
2	.02500	5.800378	52.713300	109.9595	.462310	.0214	.0000	.0000	.02955
2	.02583	7.018473	52.734560	110.3460	.425665	.0225	.0000	.0000	.02955
3	.02667	7.217494	52.752490	110.7805	.386347	.0237	.0000	.0000	.02954
3	.02750	7.396150	52.767481	111.2849	.344599	.0252	.0000	.0000	.02954
3	.02833	7.557318	52.779898	111.8884	.300667	.0270	.0000	.0000	.02954
3	.02917	7.698011	52.790002	112.6342	.254736	.0294	.0000	.0000	.02954
3	.03000	7.799385	52.798002	113.5949	.207226	.0326	.0000	.0000	.02954
3	.03083	7.886740	52.804064	114.9086	.158173	.0373	.0000	.0000	.02954
4	.03167	7.949531	52.808305	116.8987	.107797	.0452	.0000	.0000	.02953
4	.03250	7.987363	52.810501	120.6676	.056056	.0626	.0000	.0000	.02953

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